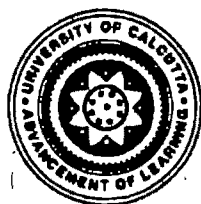
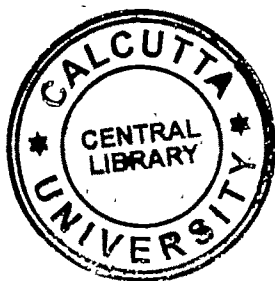




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# ON PROPERTIES OF SOME WEAK FORMS OF THE CLOSURE OPERATOR

ZBIGNIEW DUSZYNSKI

**ABSTRACT :** Characterizations of  $\alpha$ -open, semi-open, preopen, b-open, and semi-preopen subsets of an arbitrary topological space are investigated. Several formulas for  $\alpha$ -closure, semi-closure, preclosure, b-closure, and semi-preclosure of a set and their applications to the general theory of continuity of functions are given. Weak boundaries of a set are defined with respect to the above listed classes of weakly open subsets, and it is shown that three of them are always nowhere dense. In the final section we strengthen a result of Crossley and Hildebrand [C. G. Crossley, S. K. Hildebrand, *Semi-topological properties*, Fundamenta Mathematicae, 74 (1972), 233-254] concerning continuous and open functions.

**Key words and phrases :**  $\alpha$ -open, semi-open, preopen, b-open, semi-preopen sets;  $\alpha$ -closure, semi-closure, preclosure, b-closure, semi-preclosure;  $\alpha$ -irresolute, irresolute, preirresolute, b-irresolute,  $\beta$ -irresolute functions.

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## 1. PRELIMINARIES

In what follows  $(X, \tau)$  and  $(Y, \sigma)$  will denote topological spaces (briefly: spaces). Let  $S$  be a subset of a space  $(X, \tau)$ . We let  $\text{cl}(S)$  (resp.  $\text{int}(S)$ ) stand for the closure (resp. interior) of  $S$  in  $(X, \tau)$ . The set  $S$  is **regular open** (resp.  **$\alpha$ -open** [28], **semi-open** [19], **preopen** [24], **b-open** [3, 8, 14], **semi-preopen** [1, 2]) in  $(X, \tau)$  if  $S = \text{int}(\text{cl}(S))$  (resp.  $S \subset \text{int}(\text{cl}(\text{int}(S)))$ ,  $S \subset \text{cl}(\text{int}(S))$ ,  $S \subset \text{int}(\text{cl}(S))$ ,  $S \subset \text{cl}(\text{int}(S)) \cup \text{int}(\text{cl}(S))$ ,  $S \subset \text{cl}(\text{int}(\text{cl}(S)))$ ). The family of all regular open (resp.  $\alpha$ -open, semi-open, preopen, b-open, semi-preopen) subsets of a space  $(X, \tau)$  is denoted by  $\text{RO}(X, \tau)$  (resp.  $\tau^\alpha$ ,  $\text{SO}(X, \tau)$ ,  $\text{PO}(X, \tau)$ ,  $\text{BO}(X, \tau)$ ,  $\text{SPO}(X, \tau)$ ). The family  $\tau^\alpha$  forms a topology on  $X$  [28] which is weaker than  $\tau$ , in general. The following relations are known:

$$\begin{array}{ccccccc} \text{regular open} & \Rightarrow & \text{open} & \Rightarrow & \alpha\text{-open} & \Rightarrow & \text{preopen} \\ & & & & \Downarrow & & \Downarrow \\ & & & & \text{semi-open} & \Rightarrow & \text{b-open} \Rightarrow \text{semi-preopen} \end{array}$$

Let a family  $\mathcal{S} = \{S_i\}_{i \in I} \subset \text{SO}(X, \tau)$  (resp.  $\mathcal{S} \subset \text{PO}(X, \tau)$ ,  $\mathcal{S} \subset \text{BO}(X, \tau)$ ,  $\mathcal{S} \subset \text{SPO}(X, \tau)$ ) be given. Then  $\bigcup_{i \in I} S_i = \bigcup \mathcal{S} \in \text{SO}(X, \tau)$  [19] (resp.  $\bigcup \mathcal{S} \in \text{PO}(X, \tau)$  [24],  $\bigcup \mathcal{S} \in \text{BO}(X, \tau)$  [3],  $\bigcup \mathcal{S} \in \text{SPO}(X, \tau)$  [2]).

A subset  $S$  of  $(X, \tau)$  is said to be  $\alpha$ -closed [26] (resp. *semi-closed* [5], *preclosed* [24], *b-closed* [3], *semi-preclosed* [2]) if  $X \setminus S \in \tau^\alpha$  (resp.  $X \setminus S \in \text{SO}(X, \tau)$ ,  $X \setminus S \in \text{PO}(X, \tau)$ ,  $X \setminus S \in \text{BO}(X, \tau)$ ,  $X \setminus S \in \text{SPO}(X, \tau)$ ). The family of all closed (resp. semi-closed, preclosed, b-closed, semi-preclosed) subsets of  $(X, \tau)$  we denote by  $c(X, \tau)$  (resp.  $\text{SC}(X, \tau)$ ,  $\text{PC}(X, \tau)$ ,  $\text{BC}(X, \tau)$ ,  $\text{SPC}(X, \tau)$ ). The intersection of all semi-closed (resp. preclosed, b-closed, semi-preclosed) subsets of  $(X, \tau)$  containing  $S$  is called the *semi-closure* [5] (resp. *preclosure* [15], *b-closure* [3], *semi-preclosure* [2]) of  $S$  (in  $(X, \tau)$ ) and is denoted as  $\text{scl}(S)$  (resp.  $\text{pcl}(S)$ ,  $\text{bcl}(S)$ ,  $\text{spcl}(S)$ ). The union of all semi-open (resp. preopen, b-open, semi-preopen) subsets of  $(X, \tau)$  being contained in  $S$  is called the *semi-interior* (resp. *preinterior*, *b-interior*, *semi-preinterior*) of  $S$  (in  $(X, \tau)$ ) and is designated as  $\text{sint}(S)$  (resp.  $\text{pint}(S)$ ,  $\text{bint}(S)$ ,  $\text{spint}(S)$ ).

The following results are fundamental shall very useful in the sequel. Here and on,  $\text{cl}_{\tau^\alpha}(S)$  and  $\text{int}_{\tau^\alpha}(S)$  stand respectively for the closure and interior of  $S$  in the space  $(X, \tau^\alpha)$ .

**Proposition 1.** [2, Theorems 1.5, 2.15, 2.18] and [3, Proposition 2.5]. *Let  $S$  be a subset of  $(X, \tau)$ . Then:*

- ❶  $\text{cl}_{\tau^\alpha}(S) = S \cup \text{cl}(\text{int}(\text{cl}(S)))$ ,
- ❷  $\text{int}_{\tau^\alpha}(S) = S \cap \text{int}(\text{cl}(\text{int}(S)))$ ,
- ❸  $\text{scl}(S) = S \cup \text{int}(\text{cl}(S))$ ,
- ❹  $\text{sint}(S) = S \cap \text{cl}(\text{int}(S))$ ,
- ❺  $\text{pcl}(S) = S \cup \text{cl}(\text{int}(S))$ ,
- ❻  $\text{pint}(S) = S \cap \text{int}(\text{cl}(S))$ ,
- ❼  $\text{bcl}(S) = S \cup [\text{int}(\text{cl}(S)) \cap \text{cl}(\text{int}(S))]$ ,
- ❽  $\text{bint}(S) = S \cap [\text{int}(\text{cl}(S)) \cup \text{cl}(\text{int}(S))]$ ,
- ❾  $\text{spcl}(S) = S \cup \text{int}(\text{cl}(\text{int}(S)))$ ,
- ❿  $\text{spint}(S) = S \cap \text{cl}(\text{int}(\text{cl}(S)))$ .

With the above formulas we get easily the following relation ( $S$  of  $(X, \tau)$  is arbitrary):

$$\begin{aligned} \text{scl}(S) &= X \setminus \text{sint}(X \setminus S), \text{pcl}(S) = X \setminus \text{pint}(X \setminus S), \\ \text{bcl}(S) &= X \setminus \text{bint}(X \setminus S), \text{spcl}(S) = X \setminus \text{spint}(X \setminus S). \end{aligned}$$

Recall that a subfamily  $m_X \subset 2^X$ ,  $X \neq \emptyset$ , is called a *minimal structure* (briefly: *m-structure*) on  $X$  [34] if  $\emptyset \in m_X$  and  $X \in m_X$  (see also [21, 22]). A pair  $(X, m_X)$  is called then an *M-space*. An *M-space* is said to be *supratopological space* [17] if it is closed under arbitrary

union. Both  $m_X$ -closure and  $m_X$ -interior are defined in a manner analogous to those above (for  $\text{scl}(\cdot)$  etc.)

In any  $M$ -space  $(X, m_X)$  we have [21, Lemma 2.2]:

- if  $S \in m_X$  then  $m_X\text{-int}(S) = S$ , and
- if  $X \setminus S \in m_X$  then  $m_X\text{-cl}(S) = S$ .

In [34, Lemma 3.3] it has been shown that for any supratopological space  $(X, m_X)$  we have:

- if  $m_X\text{-int}(S) = S$  then  $S \in m_X$ , and
- if  $m_X\text{-cl}(S) = S$  then  $X \setminus S \in m_X$ .

Concluding,

**Proposition 2.** *For any supratopological space  $(X, m_X)$  and any  $S \subset X$  the following equivalences hold:*

- ❶  $S \in m_X$  if and only if  $m_X\text{-int}(S) = S$
- ❷  $X \setminus S \in m_X$  if and only if  $m_X\text{-cl}(S) = S$ .

Recall that in any  $M$ -space  $(X, m_X)$  the following properties hold [21, Lemma 2.3(iii)]:

- $m_X\text{-cl}(m_X\text{-cl}(S)) = m_X\text{-cl}(S)$  for every  $S \subset X$ ,
- $m_X\text{-int}(m_X\text{-int}(S)) = m_X\text{-int}(S)$  for every  $S \subset X$ .

A function  $f: (X, m_X) \rightarrow (Y, m_Y)$ , where  $(X, m_X)$  and  $(Y, m_Y)$  are  $M$ -spaces, is said to be  *$M$ -continuous* [34] if for each  $x \in X$  and each  $V \in m_Y$  containing  $f(x)$ , there exists  $U \in m_X$  containing  $x$  such that  $f(U) \subset V$ .

## 2. ON CHARACTERIZATIONS OF SOME GENERALIZED OPEN SETS

[2, Theorem 3.21] and [3, Proposition 2.1 and Remark 2] establish an interesting characterization of semi-open sets, preopen sets,  $b$ -sets, and semi-preopen sets in terms of respective closure-like and interior-like operators.

We shall show below that in the usual definitions of  $\alpha$ -open, semi-open, preopen,  $b$ -open, and semi-preopen sets, the closure operator can be replaced by the respective weak closure operator, while the interior operator stays unmodified.

We start with some lemmas.

**Lemma 1.** *Let  $(X, \tau)$  be a space and  $S \in \text{SPO}(X, \tau)$ . Then*

$$\text{cl}_\tau(S) = \text{cl}_{\tau^a}(S).$$

**Proof.** For any  $S \in \text{SPO}(X, \tau)$  we have  $\text{cl}(S) = \text{cl}(\text{int}(\text{cl}(S)))$  [33, proof of Theorem 3.1]. So, by [2, Theorem 1.5(c)] we get  $\text{cl}_{\tau^{\alpha}}(S) = S \cup \text{cl}(\text{int}(\text{cl}(S))) = \text{cl}(S)$ .

**Remark 1.** Lemma 1 improves [16, Lemma 1(i)].

**Lemma 2.** Let  $S$  be a subset of a space  $(X, \tau)$ . Then

$$\text{Int}(\text{cl}(S)) = \text{int}(\text{cl}_{\tau^{\alpha}}(S)).$$

**Proof.** Using [2, Theorem 1.5(c)] we have the inclusions  $\text{cl}(\text{int}(\text{cl}(S))) \subset \text{cl}_{\tau^{\alpha}}(S) \subset \text{cl}(S)$ . Thus, clearly,  $\text{int}(\text{cl}(S)) \subset \text{int}(\text{cl}_{\tau^{\alpha}}(S)) \subset \text{int}(\text{cl}(S))$ .

**Lemma 3.** Let  $(X, \tau)$  be a space and  $S$  a subset thereof. We have:

- ①  $\text{int}(\text{cl}(\text{int}(S))) = \text{int}(\text{cl}_{\tau^{\alpha}}(\text{int}(S)))$ ,
- ②  $\text{cl}(\text{int}(S)) = \text{cl}_{\tau^{\alpha}}(\text{int}(S))$ ,
- ③  $\text{cl}(\text{int}(\text{cl}(S))) = \text{cl}_{\tau^{\alpha}}(\text{int}(\text{cl}_{\tau^{\alpha}}(S)))$ .

**Proof.** ① We make use of Lemmas 2 and 1:

$$\text{int}(\text{cl}(\text{int}(S))) = \text{int}(\text{cl}(\text{cl}(\text{int}(S)))) = \text{int}(\text{cl}_{\tau^{\alpha}}(\text{cl}_{\tau^{\alpha}}(\text{int}(S)))) = \text{int}(\text{cl}_{\tau^{\alpha}}(\text{int}(S))).$$

- ② Obvious. ③ Using Lemmas 1 and 2 once again, we get

$$\begin{aligned} \text{cl}(\text{int}(\text{cl}(S))) &= \text{cl}(\text{int}(\text{int}(\text{cl}(S)))) = \\ &= \text{cl}_{\tau^{\alpha}}(\text{int}(\text{int}(\text{cl}_{\tau^{\alpha}}(S)))) = \text{cl}_{\tau^{\alpha}}(\text{int}(\text{cl}_{\tau^{\alpha}}(S))). \end{aligned}$$

**Theorem 1.** In any space  $(X, \tau)$  the following statements hold:

- ①  $S \in \tau^{\alpha}$  if and only if  $S \subset \text{int}(\text{cl}_{\tau^{\alpha}}(\text{int}(S)))$ ,
- ②  $S \in \text{SO}(X, \tau)$  if and only if  $S \subset \text{cl}_{\tau^{\alpha}}(\text{int}(S))$ ,
- ③  $S \in \text{PO}(X, \tau)$  if and only if  $S \subset \text{int}(\text{cl}_{\tau^{\alpha}}(S))$ ,
- ④  $S \in \text{BO}(X, \tau)$  if and only if  $S \subset \text{cl}_{\tau^{\alpha}}(\text{int}(S)) \cup \text{int}(\text{cl}_{\tau^{\alpha}}(S))$ ,
- ⑤  $S \in \text{SPO}(X, \tau)$  if and only if  $S \subset \text{cl}_{\tau^{\alpha}}(\text{int}(\text{cl}_{\tau^{\alpha}}(S)))$ .

**Proof.** Follows directly from Lemmas 2 and 3.

**Lemma 4.** Let  $(X, \tau)$  be a space and  $S$  a subset thereof. Then:

- ① [18, Proposition 2.7(a)]  $\text{scl}(\text{int}(S)) = \text{int}(\text{cl}(\text{int}(S)))$ ,

$$\textcircled{1} \text{ pcl}(\text{int}(S)) = \text{cl}(\text{int}(S)),$$

$$\textcircled{2} \text{ int}(\text{cl}(S)) = \text{int}(\text{scl}(S)).$$

**Proof.**  $\textcircled{1}$  Obvious.  $\textcircled{2}$  We use [2, Theorem 1.5(e)]:

$$\text{pcl}(\text{int}(S)) = \text{int}(S) \cup \text{cl}(\text{int}(S)) = \text{cl}(\text{int}(S)).$$

$\textcircled{3}$  By [2, Theorem 1.5(a)] we get  $\text{int}(\text{cl}(S)) \subset \text{scl}(S) \subset \text{cl}(S)$ . So, the result follows.

**Theorem 2.** *In any space  $(X, \tau)$  the following statements hold:*

$$\textcircled{1} S \in \tau^\alpha \text{ if and only if } S \subset \text{int}(\text{scl}(\text{int}(S))),$$

$$\textcircled{2} S \in \text{SO}(X, \tau) \text{ if and only if } S \subset \text{pcl}(\text{int}(S)),$$

$$\textcircled{3} S \in \text{PO}(X, \tau) \text{ if and only if } S \subset \text{int}(\text{scl}(S)),$$

$$\textcircled{4} S \in \text{BO}(X, \tau) \text{ if and only if } S \subset \text{pcl}(\text{int}(S)) \cup \text{int}(\text{scl}(S)),$$

$$\textcircled{5} S \in \text{SPO}(X, \tau) \text{ if and only if } S \subset \text{pcl}(\text{int}(\text{scl}(S))).$$

**Proof.** Cases  $\textcircled{1}$ – $\textcircled{4}$  follow directly from respective parts of Lemma 4.  $\textcircled{5}$  Observe that for any subset  $S \subset X$ , in view Lemma, 4,  $\textcircled{1}$  &  $\textcircled{2}$ , we have

$$\begin{aligned} \text{cl}(\text{int}(\text{cl}(S))) &= \text{cl}(\text{int}(\text{int}(\text{cl}(S)))) = \\ &= \text{pcl}(\text{int}(\text{int}(\text{scl}(S)))) = \text{pcl}(\text{int}(\text{scl}(S))). \end{aligned}$$

**Remark 2.** By Lemma 2 and  $\textcircled{5}$  of Lemma 3 the following alternative characterizations of semi-preopen sets follow:

$$\textcircled{5}' S \in \text{SPO}(X, \tau) \text{ if and only if } S \subset \text{pcl}(\text{int}(\text{cl}_{\tau^\alpha}(S)));$$

$$\textcircled{5}'' S \in \text{SPO}(X, \tau) \text{ if and only if } S \subset \text{cl}_{\tau^\alpha}(\text{int}(\text{scl}(S))).$$

**Remark 3.** For any subset  $S$  of a space  $(X, \tau)$ , by [2, Theorem 1.5] we get

$$\textcircled{1} \text{cl}_{\tau^\alpha}(S) = S \cup \text{cl}_{\tau^\alpha}(\text{int}(\text{cl}_{\tau^\alpha}(S))) = S \cup \text{pcl}(\text{int}(\text{cl}_{\tau^\alpha}(S))) = S \cup \text{cl}_{\tau^\alpha}(\text{int}(\text{scl}(S)));$$

$$\textcircled{2} \text{scl}(S) = S \cup \text{int}(\text{cl}_{\tau^\alpha}(S)) = S \cup \text{int}(\text{scl}(S));$$

$$\textcircled{3} \text{pcl}(S) = S \cup \text{cl}_{\tau^\alpha}(\text{int}(S)) = S \cup \text{pcl}(\text{int}(S));$$

$$\textcircled{4} \text{bcl}(S) = S \cup [\text{int}(\text{cl}_{\tau^\alpha}(S)) \cap \text{cl}_{\tau^\alpha}(\text{int}(S))] = S \cup [\text{int}(\text{scl}(S)) \cap \text{pcl}(\text{int}(S))];$$

$$\textcircled{5} \text{spcl}(S) = S \cup \text{int}(\text{cl}_{\tau^\alpha}(\text{int}(S))) = S \cup \text{scl}(\text{int}(S)).$$

All reasonings are left to the reader.

**Theorem 3.** For any space  $(X, \tau)$ ,

$$\tau^\alpha = \{S \subset X : S \subset \text{int}(\text{cl}(S)) \cap \text{cl}(\text{int}(S))\}.$$

**Proof.**  $(\Rightarrow)$   $S \in \tau^\alpha$  means  $S \subset \text{int}(\text{cl}(\text{int}(S)))$ . So,  $S \subset \text{int}(\text{cl}(S)) \cap \text{cl}(\text{int}(S))$ .  $(\Leftarrow)$  Let  $S \subset \text{int}(\text{cl}(S)) \cap \text{cl}(\text{int}(S))$ . Thus  $S \in \text{SO}(X, \tau) \cap \text{PO}(X, \tau) = \tau^\alpha$  [32, Lemma 3.1].

**Corollary 1.** In any space  $(X, \tau)$ ,

$$S \in \text{c}(X, \tau^\alpha) \text{ if and only if } \text{int}(\text{cl}(S)) \cup \text{cl}(\text{int}(S)) \subset S.$$

It is obvious, that  $\text{scl}(S) \cup \text{pcl}(S) \subset \text{cl}_{\tau^\alpha}(S)$  for any  $S \subset X$ . In view of the example below, this inclusion can be proper.

**Example 1.** Consider the real line  $(\mathbb{R}, \tau_e)$  equipped with the Euclidean topology and  $S = \mathbb{Q} \cap (1, 2)$ , where  $\mathbb{Q}$  is the set of all rationals.

**Remark 4.** By Example 1 and by [2, Theorem 1.5(a), (e)] one gets that the formula  $\text{cl}_{\tau^\alpha}(S) = S \cup \text{int}(\text{cl}(S)) \cup \text{cl}(\text{int}(S))$  does not hold for all  $S \subset X$ .

In order to obtain a characterization for b-open sets, similar to that of [2, Theorem 3.21] and [3, Proposition 2.1(c) and Remark 2] (given for semi-open, preopen, b-open, and semi-preopen sets), we have to recall some results.

**Lemma 5.** Let  $(X, \tau)$  be arbitrary.

- (a) [32, Lemma 3.5] and [11]. If either  $S_1 \in \text{SO}(X, \tau) \cup \text{SC}(X, \tau)$  or  $S_2 \in \text{SO}(X, \tau) \cup \text{SC}(X, \tau)$ , then
  - (a<sub>1</sub>)  $\text{int}(\text{cl}(S_1 \cap S_2)) = \text{int}(\text{cl}(S_1)) \cap \text{int}(\text{cl}(S_2))$ ,
  - (a<sub>2</sub>)  $\text{cl}(\text{int}(S_1 \cup S_2)) = \text{cl}(\text{int}(S_1)) \cup \text{cl}(\text{int}(S_2))$ .
- (b) [12]. If  $S_1, S_2 \subset X$ , then
  - (b<sub>1</sub>)  $\text{int}(\text{cl}(S_1 \cup S_2)) = \text{int}(\text{cl}(\text{int}(\text{cl}(S_1)) \cup \text{int}(\text{cl}(S_2))))$ ,
  - (b<sub>2</sub>)  $\text{cl}(\text{int}(S_1 \cap S_2)) = \text{cl}(\text{int}(\text{cl}(\text{int}(S_1)) \cup \text{cl}(\text{int}(S_2))))$ .

**Corollary 2.** Let  $S_1, S_2$  be arbitrary subsets of a space  $(X, \tau)$ . Then

- (a<sub>3</sub>)  $\text{int}(\text{cl}(\text{int}(S_1 \cap S_2))) = \text{int}(\text{cl}(\text{int}(S_1))) \cap \text{int}(\text{cl}(\text{int}(S_2)))$ ,
- (a<sub>4</sub>)  $\text{cl}(\text{int}(\text{cl}(S_1 \cup S_2))) = \text{cl}(\text{int}(\text{cl}(S_1))) \cup \text{cl}(\text{int}(\text{cl}(S_2)))$ .

**Proof.** (a<sub>3</sub>) Apply Lemma 5(a<sub>1</sub>) to the sets  $\text{int}(S_1)$  and  $\text{int}(S_2)$ . (a<sub>4</sub>) is dual to (a<sub>3</sub>).

**Lemma 6.** Let  $(X, \tau)$  be a space and  $S$  a subset thereof. Then

- ❶  $\text{bcl}(S) = S \cup \text{bcl}(\text{bint}(S))$ ,
- ❷  $\text{bcl}(S) = S \cup \text{bint}(\text{bcl}(S))$ .



**Proof.** ❶ Using Lemma 5(a<sub>1</sub>), (b<sub>2</sub>) we calculate as follows:

$$\begin{aligned} \text{bcl}(\text{bint}(S)) &= \text{bcl}(S \cap (\text{int}(\text{cl}(S)) \cup \text{cl}(\text{int}(S)))) = \text{bint}(S) \cup \\ &\cup [\text{int}(\text{cl}(S \cap (\text{int}(\text{cl}(S)) \cup \text{cl}(\text{int}(S)))) \cap \text{cl}(\text{int}(S \cap (\text{int}(\text{cl}(S)) \cup \text{cl}(\text{int}(S)))))] = \\ &= \text{bint}(S) \cup [\text{int}(\text{cl}(S)) \cap \text{cl}(\text{int}(\text{cl}(\text{int}(S)))) \cap \text{cl}(\text{int}(\text{int}(\text{cl}(S)) \cup \text{cl}(\text{int}(S))))]. \end{aligned}$$

(recall that  $\text{int}(\text{cl}(S)), \text{cl}(\text{int}(S)) \in \text{SO}(X, \tau)$ , whence  $\text{int}(\text{cl}(S)) \cup \text{cl}(\text{int}(S)) \in \text{SO}(X, \tau)$ , since  $(X, \text{SO}(X, \tau))$  is a supratopological space, see [19, Theorem 2]). On the other hand, the sets  $\text{int}(\text{cl}(S)), \text{cl}(\text{int}(S)) \in \text{SC}(X, \tau)$  whence by Lemma 5(a<sub>2</sub>) we get

$$\text{cl}[\text{int}[\text{cl}(\text{int}(S)) \cap \text{cl}(\text{int}(\text{int}(\text{cl}(S)) \cup \text{cl}(\text{int}(S))))]] = \text{cl}(\text{int}(S)).$$

Thus we have

$$\text{bcl}(\text{bint}(S)) = \text{bint}(S) \cup (\text{int}(\text{cl}(S)) \cap \text{cl}(\text{int}(S)))$$

and finally

$$S \cup \text{bcl}(\text{bint}(S)) = \text{bint}(S) \cup \text{bcl}(S) = \text{bcl}(S).$$

❷ It follows directly from the fact that  $\text{bcl}(\text{bint}(S)) = \text{bint}(\text{bcl}(S))$  for any  $S \subset X$  [3, Proposition 2.7].

**Theorem 4.** *Let  $(X, \tau)$  be a space and  $S$  a subset thereof. Then, the following statements are equivalent:*

- ❶  $S \in \text{BC}(X, \tau)$ ,
- ❷  $\text{bcl}(\text{bint}(S)) \subset S$ ,
- ❸  $\text{bint}(\text{bcl}(S)) \subset S$ .

**Proof.** ❶ $\Leftrightarrow$ ❷ Since  $(X, \text{BO}(X, \tau))$  is a supratopological space, see [3, Proposition 2.3(a)], by ❶ from Proposition 2 we have that  $S \in \text{BC}(X, \tau)$  if and only if  $\text{bcl}(S) = S$ . So, the result follows directly by ❶ of Lemma 6. ❷ $\Leftrightarrow$ ❸ Use [3, Proposition 2.7].

**Corollary 3.** *Let  $(X, \tau)$  be a space and  $S$  a subset thereof. The following statements are equivalent:*

- ❶  $S \in \text{BO}(X, \tau)$ ,
- ❷  $\text{bcl}(\text{bint}(S)) \supset S$ ,
- ❸  $\text{bint}(\text{bcl}(S)) \supset S$ .

**Proof.** Clear.

### 3. WEAK CLOSURES, WEAK INTERIORS AND WEAK BOUNDARIES

We start this section with results similar to those in Lemma 6, but related to the another closure-like operators which are studied in the paper.

**Proposition 3.** *Let  $S$  be a subset of  $(X, \tau)$ . Then the following hold:*

- ①  $\text{scl}(S) = S \cup \text{sint}(\text{scl}(S))$ ,
- ②  $\text{pcl}(S) = S \cup \text{pcl}(\text{pint}(S))$ ,
- ③  $\text{bcl}(S) = S \cup \text{pint}(\text{pcl}(S))$ ,
- ④  $\text{spcl}(S) = S \cup \text{scl}(\text{sint}(S)) = S \cup \text{spcl}(\text{spint}(S)) = S \cup \text{spint}(\text{spcl}(S))$ .

**Proof.** ① By ② & ④ of Proposition 1 and by Lemma 5(a<sub>2</sub>) we have

$$\text{sint}(\text{scl}(S)) = \text{scl}(S) \cap \text{cl}(\text{int}(S \cup \text{int}(\text{cl}(S)))) = \text{scl}(S) \cap \text{cl}(\text{int}(\text{cl}(S))).$$

So,

$$S \cup \text{sint}(\text{scl}(S)) = \text{scl}(S) \cap \text{cl}_{\tau^a}(S) = \text{scl}(S).$$

② Using ③ & ④ of Proposition 1 we get

$$\begin{aligned} S \cup \text{pcl}(\text{pint}(S)) &= S \cup \text{pint}(S) \cup \text{cl}(\text{int}(S \cap \text{int}(\text{cl}(S)))) = \\ &= S \cup \text{pint}(S) \cup \text{cl}(\text{int}(S)) = \text{pint}(S) \cup \text{pcl}(S) = \text{pcl}(S). \end{aligned}$$

③ We make use of Proposition 1 ③, Theorem 7④, and [3, Proposition 2.5(1)]. Then

$$\begin{aligned} S \cup \text{pint}(\text{pcl}(S)) &= S \cup (\text{pcl}(S) \cap \text{int}(\text{cl}(\text{pcl}(S)))) = \\ &= \text{pcl}(S) \cap (S \cup \text{int}(\text{cl}(S))) = \text{pcl}(S) \cap \text{scl}(S) = \text{bcl}(S). \end{aligned}$$

④ (compare [2, Corollary 3.19(a)]) Since  $\text{int}(\text{spint}(S)) = \text{int}(S)$  (dual to  $\text{cl}(\text{spcl}(S)) = \text{cl}(S)$ , see Theorem 7), we have

$$\begin{aligned} S \cup \text{spcl}(\text{spint}(S)) &= S \cup \text{spint}(S) \cup \text{int}(\text{cl}(\text{int}(\text{spint}(S)))) = \\ &= S \cup \text{spint}(S) \cup \text{int}(\text{cl}(\text{int}(S))) = S \cup \text{int}(\text{cl}(\text{int}(S))) = \text{spcl}(S). \end{aligned}$$

By [2, Theorem 3.18] we get  $\text{spcl}(S) = S \cup \text{spint}(\text{spcl}(S))$ . The third equality we leave to the reader.

Arguing similarly as in Theorem 4 and Corollary 3, using respectively ①, ②, ③, ④ of the above proposition, one can characterize members of respectively:  $\text{SC}(X, \tau)$  and  $\text{SO}(X, \tau)$ ,  $\text{PC}(X, \tau)$  and  $\text{PO}(X, \tau)$ ,  $\text{BC}(X, \tau)$  and  $\text{BO}(X, \tau)$ ,  $\text{SPC}(X, \tau)$  and  $\text{SPO}(X, \tau)$ . These results are due to andrijevic – the reader is advised to compare [3, Remark 2 & Proposition 2.1] and [2, Theorem 3.21]. The formula  $\text{cl}_{\tau^a}(S) = S \cup \text{cl}_{\tau^a}(\text{int}_{\tau^a}(\text{cl}_{\tau^a}(S)))$ ,  $S \subset X$ , follows by [18,

Corollary 2.4(b)]. Moreover, with [18, Corollary 2.4(a)], its dual, and the dual to [18, Corollary 2.4(b)] one gets yet another formulas (here  $S \subset X$  is arbitrary):

$$(a) \text{ scl}(S) = S \cup \text{int}_{\tau^\alpha}(\text{cl}_{\tau^\alpha}(S)) = \text{scl}_{\tau^\alpha}(S),$$

$$(b) \text{ pcl}(S) = S \cup \text{cl}_{\tau^\alpha}(\text{int}_{\tau^\alpha}(S)) = \text{pcl}_{\tau^\alpha}(S),$$

$$(c) \text{ bcl}(S) = S \cup [\text{int}_{\tau^\alpha}(\text{cl}_{\tau^\alpha}(S)) \cap \text{cl}_{\tau^\alpha}(\text{int}_{\tau^\alpha}(S))] = \text{bcl}_{\tau^\alpha}(S),$$

$$(d) \text{ spcl}(S) = S \cup \text{int}_{\tau^\alpha}(\text{cl}_{\tau^\alpha}(\text{int}_{\tau^\alpha}(S))) = \text{spcl}_{\tau^\alpha}(S),$$

where  $\text{scl}_{\tau^\alpha}$ ,  $\text{pcl}_{\tau^\alpha}$ ,  $\text{bcl}_{\tau^\alpha}$ , and  $\text{spcl}_{\tau^\alpha}$  are respective closure-like operators defined with respect to the topology  $\tau^\alpha$  in  $X$ .

Under passage to the complement, it is easy to see that for any subset  $S$  of  $(X, \tau)$ :  $\text{sint}_{\tau^\alpha}(S) = \text{sint}(S)$ ,  $\text{pint}_{\tau^\alpha}(S) = \text{pint}(S)$ ,  $\text{bint}_{\tau^\alpha}(S) = \text{bint}(S)$ ,  $\text{spint}_{\tau^\alpha}(S) = \text{spint}(S)$ . So, we get a modified version of Proposition 3.

**Proposition 3'.** *Let  $S$  be a subset of  $(X, \tau)$ . Then the following hold:*

$$\textcircled{1} \text{ scl}(S) = S \cup \text{sint}_{\tau^\alpha}(\text{scl}_{\tau^\alpha}(S)),$$

$$\textcircled{2} \text{ pcl}(S) = S \cup \text{pcl}_{\tau^\alpha}(\text{pint}_{\tau^\alpha}(S)),$$

$$\textcircled{3} \text{ bcl}(S) = S \cup \text{pint}_{\tau^\alpha}(\text{pcl}_{\tau^\alpha}(S)),$$

$$\textcircled{4} \text{ spcl}(S) = S \cup \text{scl}_{\tau^\alpha}(\text{sint}_{\tau^\alpha}(S)) = S \cup \text{spcl}_{\tau^\alpha}(\text{spint}_{\tau^\alpha}(S)) = S \cup \text{spint}_{\tau^\alpha}(\text{spcl}_{\tau^\alpha}(S)).$$

The following elementary property is well-known:  $\text{cl}(\text{int}(\text{cl}(\text{int}(S)))) = \text{cl}(\text{int}(S))$  (dually  $\text{int}(\text{cl}(\text{int}(\text{cl}(S)))) = \text{int}(\text{cl}(S))$ ) for any subset  $S$  of a space  $(X, \tau)$ . We shall show that analogous properties hold for weak forms of closure and interior operators.

**Theorem 5.** *Let  $S$  be a subset of  $(X, \tau)$ . Then:*

$$\textcircled{1} \text{ scl}(\text{sint}(\text{scl}(\text{sint}(S)))) = \text{scl}(\text{sint}(S)),$$

$$\textcircled{2} \text{ pcl}(\text{pint}(\text{pcl}(\text{pint}(S)))) = \text{pcl}(\text{pint}(S)),$$

$$\textcircled{3} \text{ bcl}(\text{bint}(\text{bcl}(\text{bint}(S)))) = \text{bcl}(\text{bint}(S)),$$

$$\textcircled{4} \text{ spcl}(\text{spint}(\text{spcl}(\text{spint}(S)))) = \text{spcl}(\text{spint}(S)).$$

**Proof.**  $\textcircled{1}$  By [2, Corollary 3.4] we have

$$\text{sint}(\text{scl}(\text{sint}(S))) = \text{scl}(\text{sint}(S)).$$

Since  $\text{scl}(\text{scl}(A)) = \text{scl}(A)$  for any  $A \subset X$  [5, Theorem 1.7(3)] (or by [21, Lemma 2.3(iii)]), the result follows.

② By [2, Corollary 3.8(c), (b)] we obtain

$$\text{pcl}(\text{pint}(\text{pcl}(\text{pint}(S)))) = \text{pcl}(\text{pint}(\text{pcl}(S))) = \text{pcl}(\text{pint}(S)).$$

③ Use the identity  $\text{bint}(\text{bcl}(S)) = \text{bcl}(\text{bint}(S))$  (see [3, Proposition 2.7]) and [21, Lemma 2.3(iii)].

④ Apply the identity  $\text{spint}(\text{spcl}(S)) = \text{spcl}(\text{spint}(S))$ ,  $S \subset X$  (see [2, Theorem 3.18]) and [21, Lemma 2.3(iii)].

We define now certain weak boundary operators.

**Definition 1.** Let  $S$  be a subset of a space  $(X, \tau)$ . We set

- the *semi-boundary* of  $S$  as  $\text{sFr}(S) = \text{scl}(S) \setminus \text{sint}(S)$ ,
- the *preboundary* of  $S$  as  $\text{pFr}(S) = \text{pcl}(S) \setminus \text{pint}(S)$ ,
- the *b-boundary* of  $S$  as  $\text{bFr}(S) = \text{bcl}(S) \setminus \text{bint}(S)$ ,
- the *semi-preboundary* of  $S$  as  $\text{spFr}(S) = \text{spcl}(S) \setminus \text{spint}(S)$ .

Moreover,  $\alpha\text{Fr}(S)$  will stand for the boundary of  $S$  in the space  $(X, \tau^\alpha)$ .

**Remark 5.** Let  $(X, \tau)$  be a space,  $S \subset X$ . Then

- ①  $\text{spFr}(S) \subset \text{bFr}(S) \subset \text{sFr}(S) \subset \alpha\text{Fr}(S) \subset \text{Fr}(S)$ ,
- ②  $\text{bFr}(S) \subset \text{pFr}(S) \subset \alpha\text{Fr}(S)$ .

There is no genenral inclusion-relaitonship between  $\text{sFr}(S)$  and  $\text{pFr}(S)$ .

**Example 2.** Consider  $(\mathbb{R}, \tau_c)$  and the set  $S = (0, 1) \cup (\mathbb{Q} \cap (1, 2))$ . One checks that  $\text{sFr}(S) = [1, 2)$  and  $\text{pFr}(S) = \{0, 1\}$ .

Since  $S \in \text{SO}(X, \tau)$  if and only if  $\text{cl}(S) = \text{cl}(\text{int}(S))$  [29, Lemma 2] and, dually,  $S \in \text{SC}(X, \tau)$  if and only if  $\text{int}(S) = \text{int}(\text{cl}(S))$ , we obtain that all above boundaries are nowhere dense for  $S \in \text{SO}(X, \tau) \cup \text{SC}(X, \tau)$ . For  $\text{pFr}(S)$ ,  $\text{bFr}(S)$ , and  $\text{spFr}(S)$  this statement shall be generalized in the sequel.

**Proposition 4.** Let  $S$  be a subset of  $(X, \tau)$ . We have:

- ①  $\text{int}(\text{cl}(\alpha\text{Fr}(S))) = \text{int}(\text{cl}(S)) \cap \text{int}(\text{cl}(X \setminus S))$ ,
- ②  $\text{int}(\text{cl}(\text{sFr}(S))) = \text{int}(\text{cl}(\alpha\text{Fr}(S)))$ .

**Proof.** ① By definition of  $\alpha\text{Fr}(S)$  and by Lemma 5, (a<sub>1</sub>) & (b<sub>1</sub>), we have what follows:

$$\begin{aligned} \text{int}(\text{cl}(\alpha\text{Fr}(S))) &= \text{int}[\text{cl}[S \cup \text{cl}(\text{int}(\text{cl}(S)))] \cap ((X \setminus S) \cup \text{cl}(\text{int}(\text{cl}(X \setminus S))))] = \\ &= \text{int}[\text{cl}[\text{int}(\text{cl}(S \cap \text{cl}(\text{int}(\text{cl}(X \setminus S))))]] \cup [\text{int}(\text{cl}((X \setminus S) \cap \text{cl}(\text{int}(\text{cl}(S)))) \cup ] \end{aligned}$$

$$\begin{aligned} & \cup [\text{int}(\text{cl}(\text{cl}(\text{int}(\text{cl}(S))) \cap \text{cl}(\text{int}(\text{cl}(X \setminus S)))))] = \\ & = \text{int}(\text{cl}(\text{int}(\text{cl}(S)) \cap \text{int}(\text{cl}(X \setminus S)))) = \text{int}(\text{cl}(S)) \cap \text{int}(\text{cl}(X \setminus S)), \end{aligned}$$

because the intersection of any two regular open sets is regular open [9, Problem 22(g)].

② By the definition of  $\text{sFr}(S)$ , Lemma 5(a<sub>1</sub>), and since  $\text{scl}(S) \in \text{SC}(X, \tau)$  [5, Remark 1.5] (compare also ② of Proposition 2 and [21, Lemma 2.3(iii)] for  $m_X = \text{SO}(X, \tau)$ ) we obtain

$$\text{int}(\text{cl}(\text{sFr}(S))) = \text{int}(\text{cl}(S \cup \text{int}(\text{cl}(S)))) \cap \text{int}(\text{cl}((X \setminus S) \cup \text{int}(\text{cl}(X \setminus S)))).$$

So, making use of Lemma 5(b<sub>1</sub>) one gets

$$\text{int}(\text{cl}(\text{sFr}(S))) = \text{int}(\text{cl}(S)) \cap \text{int}(\text{cl}(X \setminus S)).$$

**Remark 6.** If  $S \notin \text{SO}(X, \tau) \cup \text{SC}(X, \tau)$ , then  $\alpha\text{Fr}(S)$  and  $\text{sFr}(S)$  need not be nowhere dense in  $(X, \tau)$ . It is enough to consider the set  $S$  from Example 1 and to apply Proposition 4.

**Theorem 6.** Let  $(X, \tau)$  be a space,  $S \subset X$ . Each of the sets  $\text{pFr}(S)$ ,  $\text{bFr}(S)$ , and  $\text{spFr}(S)$  is nowhere dense in  $(X, \tau)$ .

**Proof.** By Remark 5 it is enough to prove that  $\text{int}(\text{cl}(\text{pFr}(S))) = \emptyset$ . By definition of  $\text{pFr}(S)$  and by Lemma 5(b<sub>1</sub>) we have

$$\begin{aligned} \text{int}(\text{cl}(\text{pFr}(S))) &= \text{int}[\text{cl}[(S \cup \text{cl}(\text{int}(S))) \cap ((X \setminus S) \cup \text{cl}(\text{int}(X \setminus S)))] = \\ &= \text{int}[\text{cl}[\text{int}(\text{cl}(S \cap \text{cl}(\text{int}(X \setminus S)))) \cup \text{int}(\text{cl}((X \setminus S) \cap \text{cl}(\text{int}(S)))) \cup \\ &\quad \cup \text{int}(\text{cl}(\text{cl}(\text{int}(S)) \cap \text{cl}(\text{int}(X \setminus S)))]]. \end{aligned}$$

But, every set  $\text{cl}(\text{int}(A)) \in \text{SO}(X, \tau)$ , thus making use of Lemma 5(a<sub>1</sub>) and Corollary 2(a<sub>3</sub>) we get

$$\begin{aligned} \text{int}(\text{cl}(\text{pFr}(S))) &= \text{int}[\text{cl}[(\text{int}(\text{cl}(S)) \cap \text{int}(\text{cl}(\text{int}(X \setminus S)))) \cup \\ &\quad \cup (\text{int}(\text{cl}(X \setminus S)) \cap \text{int}(\text{cl}(\text{int}(S)))) \cup (\text{int}(\text{cl}(\text{int}(S))) \cap \text{int}(\text{cl}(\text{int}(X \setminus S)))] = \\ &= \text{int}[\text{cl}[\text{int}(\text{cl}(S \cap \text{int}(X \setminus S))) \cup \\ &\quad \cup \text{int}(\text{cl}((X \setminus S) \cap \text{int}(S))) \cup \text{int}(\text{cl}(\text{int}((X \setminus S) \cap S)))] = \emptyset. \end{aligned}$$

The proof is complete.

Directly from Definition 1 and Proposition 2 ② we infer the following

**Corollary 4.** For arbitrary  $(X, \tau)$  and  $S \subset X$  we have:

- ①  $S \in c(X, \tau^a)$  if and only if  $\alpha\text{Fr}(S) \subset S$ ,
- ②  $S \in \text{SC}(X, \tau)$  if and only if  $\text{sFr}(S) \subset S$ ,
- ③  $S \in \text{PC}(X, \tau)$  if and only if  $\text{pFr}(S) \subset S$ ,

④  $S \in BC(X, \tau^\alpha)$  if and only if  $bFr(S) \subset S$ ,

⑤  $S \in SPC(X, \tau^\alpha)$  if and only if  $spFr(S) \subset S$ .

At the end of this section we complete results from Lemma 5 and Corollary 2.

**Proposition 5.** *Let  $(X, \tau)$  be arbitrary. If either  $S_1 \in SO(X, \tau) \cup SC(X, \tau)$  or  $S_2 \in SO(X, \tau) \cup SC(X, \tau)$ , then*

$$(a_5) \text{ int}(\text{cl}(\text{int}(S_1 \cup S_2))) = \text{int}(\text{cl}(\text{int}(S_1) \cup \text{int}(S_2))),$$

$$(a_6) \text{ cl}(\text{int}(\text{cl}(S_1 \cap S_2))) = \text{cl}(\text{int}(\text{cl}(S_1) \cap \text{cl}(S_2))).$$

**Proof.**  $(a_5)$  Using Lemma 5,  $(a_2)$  &  $(b_1)$ , we calculate as follows:

$$\begin{aligned} \text{int}(\text{cl}(\text{int}(S_1 \cup S_2))) &= \text{int}(\text{cl}(\text{cl}(\text{int}(S_1 \cup S_2)))) = \text{int}(\text{cl}(\text{cl}(\text{int}(S_1)) \cup \text{cl}(\text{int}(S_2)))) = \\ &= \text{int}(\text{cl}(\text{int}(\text{cl}(\text{int}(S_1))) \cup \text{int}(\text{cl}(\text{int}(S_2))))) = \\ &= \text{int}(\text{cl}(\text{int}(S_1) \cup \text{int}(S_2))). \end{aligned}$$

$(a_6)$  Dual to  $(a_5)$ .

Even if both  $S_1, S_2 \in PO(X, \tau)$ , the identities  $(a_1)$  from Lemma 5 and  $(a_5)$  from Proposition 5 can fail.

**Example 3.** Consider  $(\mathbb{R}, \tau_e)$  with  $S_1 = \mathbb{Q} \cap (1, 2)$ ,  $S_2 = (1, 2) \setminus \mathbb{Q}$ .

#### 4. SYMMETRICITY PROPERTIES

One can easily prove the following

**Theorem 7.** *For any  $(X, \tau)$  and  $S \subset X$ ,*

$$\textcircled{1} \text{ cl}(S) = \text{cl}(\text{cl}_{\tau^\alpha}(S)) = \text{cl}(\text{pcl}(S)) = \text{cl}(\text{scl}(S)) = \text{cl}(\text{bcl}(S)) = \text{cl}(\text{spcl}(S)),$$

$$\textcircled{2} \text{ int}(\text{cl}(S)) = \text{int}(\text{cl}(\text{pint}(S))) = \text{int}(\text{cl}(\text{bint}(S))) = \text{int}(\text{cl}(\text{spint}(S))).$$

**Proof.**  $\textcircled{1}$  Use Proposition 1.  $\textcircled{2}$  We apply Definition 1, Lemma 5(b<sub>1</sub>), and Theorem 6.

**Theorem 8.** *For any  $(X, \tau)$  and  $S \subset X$ ,*

$$\textcircled{1} \text{ cl}_{\tau^\alpha}(\text{pcl}(S)) = \text{cl}_{\tau^\alpha}(S) = \text{pcl}(\text{cl}_{\tau^\alpha}(S)),$$

$$\textcircled{2} \text{ cl}_{\tau^\alpha}(\text{scl}(S)) = \text{cl}_{\tau^\alpha}(S) = \text{scl}(\text{cl}_{\tau^\alpha}(S)),$$

$$\textcircled{3} \text{ cl}_{\tau^\alpha}(\text{bcl}(S)) = \text{cl}_{\tau^\alpha}(S) = \text{bcl}(\text{cl}_{\tau^\alpha}(S)),$$

$$\textcircled{4} \text{ cl}_{\tau^\alpha}(\text{spcl}(S)) = \text{cl}_{\tau^\alpha}(S) = \text{spcl}(\text{cl}_{\tau^\alpha}(S)).$$

**Proof.** We shall prove only the left-hand identities in ①-④, because the right-hand ones follow directly from Proposition 2 ② and the fact that for any  $S$ ,  $cl_{\tau\alpha}(S) \in c(X, \tau^\alpha) = SC(X, \tau) \cap PC(X, \tau)$  (the dual to [32, Lemma 3.1]).

① Using [2, Theorem 1.5(c), (e)] we have

$$\begin{aligned} cl_{\tau\alpha}(pcl(S)) &= pcl(S) \cup cl(int(cl(pcl(S)))) = \\ &= pcl(S) \cup cl(int(cl(S))) = [S \cup cl(int(S))] \cup cl(int(cl(S))) = cl_{\tau\alpha}(S). \end{aligned}$$

② By [2, Theorem 1.5(c), (a)] we get

$$\begin{aligned} cl_{\tau\alpha}(scl(S)) &= scl(S) \cup cl(int(cl(S \cup int(cl(S)))) = \\ &= scl(S) \cup cl(int(cl(S))) = cl_{\tau\alpha}(S). \end{aligned}$$

③ We use [2, Theorem 1.5(c)] and Proposition 1, ① & ②. We have

$$\begin{aligned} cl_{\tau\alpha}(bcl(S)) &= bcl(S) \cup cl(int(cl(S \cup (int(cl(S)) \cap cl(int(S))))) = \\ &= bcl(S) \cup cl(int(cl(S) \cup cl(int(cl(S)) \cap cl(int(S)))). \end{aligned}$$

But

$$cl(int(cl(S)) \cap cl(int(S))) \subset cl(int(cl(S))) \subset cl(S),$$

so

$$cl_{\tau\alpha}(bcl(S)) = [S \cup (int(cl(S)) \cap cl(int(S)))] \cup cl(int(cl(S))) = cl_{\tau\alpha}(S).$$

④ Left to the reader.

**Theorem 9.** Let  $(X, \tau)$  be a space,  $S \subset X$ . Then

$$\textcircled{1} \quad scl(bcl(S)) = scl(S) = bcl(scl(S)),$$

$$\textcircled{2} \quad scl(spcl(S)) = scl(S) = spcl(scl(S)).$$

**Proof.** Since  $scl(S) \in SC(X, \tau)$  [5, Remark 1.5], the right-hand equalities in ① and ② follow by Proposition 2 ②.

① By [2, Theorem 1.5(a)] and Proposition 1, ② & ③, we have what follows:

$$\begin{aligned} scl(bcl(S)) &= bcl(S) \cup int(cl[S \cup (int(cl(S)) \cap cl(int(S))]) = \\ &= bcl(S) \cup int[cl(S) \cup cl(int(cl(S)) \cap cl(int(S)))] = \\ &= bcl(S) \cup int(cl(S)) = scl(S). \end{aligned}$$

② Making use of [2, Theorems 1.5(a) & 2.15] we obtain

$$scl(spcl(S)) = spcl(S) \cup int[cl(S) \cup cl(int(S))] = spcl(S) \cup int(cl(S)).$$

**Theorem 10.** *Let  $(X, \tau)$  be a space,  $S \subset X$ . Then*

$$\textcircled{1} \text{ pcl}(\text{bcl}(S)) = \text{pcl}(S) = \text{bcl}(\text{pcl}(S)),$$

$$\textcircled{2} \text{ pcl}(\text{spcl}(S)) = \text{pcl}(S) = \text{spcl}(\text{pcl}(S)).$$

**Proof.** For  $S \subset X$  we have  $X \setminus \text{pcl}(S) = \text{pint}(X \setminus S)$ , hence  $X \setminus \text{pcl}(S) \in \text{PO}(X, \tau)$  (the union of any family of preopen sets is preopen). So,  $\text{pcl}(S) \in \text{PC}(X, \tau)$  and by Proposition 2  $\textcircled{2}$  the right-hand identities in  $\textcircled{1}$  and  $\textcircled{2}$  follow.

$\textcircled{1}$  Applying [2, Theorem 1.5(e)] and Proposition 1,  $\textcircled{3}$  &  $\textcircled{4}$ , and Lemma 5(a<sub>2</sub>), we calculate:

$$\begin{aligned} \text{pcl}(\text{bcl}(S)) &= \text{bcl}(S) \cup \text{cl}(\text{int}[S \cup (\text{int}(\text{cl}(S)) \cap \text{cl}(\text{int}(S))]) = \\ &= \text{bcl}(S) \cup [\text{cl}(\text{int}(S)) \cup \text{cl}(\text{int}(\text{cl}(S)) \cap \text{int}(\text{cl}(\text{int}(S)))] = \\ &= \text{bcl}(S) \cup \text{cl}(\text{int}(S)) = \text{pcl}(S). \end{aligned}$$

$\textcircled{2}$  The proof is similar to that for  $\textcircled{1}$  and hence omitted.

**Theorem 11.** *Let  $(X, \tau)$  be a space,  $S \subset X$ . Then*

$$\text{bcl}(\text{spcl}(S)) = \text{bcl}(S) = \text{spcl}(\text{bcl}(S)).$$

**Proof.** For any  $S \subset X$ ,  $X \setminus \text{bcl}(S) = \text{bint}(X \setminus S)$ , whence by [3, Proposition 2.3(a) and Definition 3] we get that  $\text{bcl}(S) \in \text{BC}(X, \tau) \subset \text{SPC}(X, \tau)$ . So, from Proposition 2  $\textcircled{2}$  we obtain  $\text{bcl}(S) = \text{spcl}(\text{bcl}(S))$ . In order to prove the second identity we apply [3, Proposition 2.5(1)]:  $\text{bcl}(S) = \text{scl}(S) \cap \text{pcl}(S)$ . By Theorems 9  $\textcircled{2}$  and 10  $\textcircled{2}$  we have

$$\text{bcl}(\text{spcl}(S)) = \text{scl}(\text{spcl}(S)) \cap \text{pcl}(\text{spcl}(S)) = \text{scl}(S) \cap \text{pcl}(S) = \text{bcl}(S).$$

Andrijević has proved in [2, Theorem 3.15] that (a)  $\text{pcl}(\text{scl}(S)) = \text{cl}_{\tau^*}(S)$  and (b)  $\text{scl}(\text{pcl}(S)) = \text{pcl}(S) \cup \text{scl}(S)$  hold for arbitrary  $S$ . In [2, Example 3.17] he observed that, in general,  $\text{pcl}(\text{scl}(S)) \neq \text{scl}(\text{pcl}(S))$  (see Remark 4).

Combining the left-hand identities in Theorems 9-11 and formulas  $\textcircled{1}$ – $\textcircled{3}$  of Proposition 3 (or of Proposition 3') we arrive at other (non-trivial) formulas for  $\text{scl}(S)$ ,  $\text{pcl}(S)$ , and  $\text{bcl}(S)$ . Namely, we have

**Proposition 6.** *Consider any  $(X, \tau)$ ,  $S \subset X$ . We have*

$$\textcircled{1} \text{ scl}(S) = \text{bcl}(S) \cup \text{sint}(\text{scl}(S)),$$

$$\textcircled{1} \text{ scl}(S) = \text{spcl}(S) \cup \text{sint}(\text{scl}(S)),$$

$$\textcircled{2} \text{ pcl}(S) = \text{bcl}(S) \cup \text{pcl}(\text{pint}(\text{bcl}(S))),$$

$$\textcircled{2} \text{ pcl}(S) = \text{spcl}(S) \cup \text{pcl}(\text{pint}(\text{spcl}(S))),$$

$$\textcircled{3} \text{ bcl}(S) = \text{spcl}(S) \cup \text{pint}(\text{pcl}(S)).$$



**Proof.** We prove only ③. With the aid of Theorem 10, Proposition 3 ③, and Theorem 10 ②, we calculate:

$$\text{bcl}(S) = \text{bcl}(\text{spcl}(S)) = \text{spcl}(S) \cup \text{pint}(\text{pcl}(\text{spcl}(S))) = \text{spcl}(S) \cup \text{pint}(\text{pcl}(S)).$$

**Theorem 12.** Consider any  $(X, \tau)$ ,  $S \subset X$ . We have

$$\textcircled{1} \text{ scl}(\text{pcl}(\text{scl}(S))) = cl_{\tau\alpha}(S) = \text{pcl}(\text{scl}(\text{pcl}(S))),$$

$$\textcircled{2} \text{ pcl}(\text{scl}(\text{pcl}(\text{scl}(S)))) = cl_{\tau\alpha}(S) = \text{scl}(\text{pcl}(\text{scl}(\text{pcl}(S)))).$$

**Proof.** ① By [2, Theorem 3.15] and Theorem 8 ② we obtain  $\text{scl}(\text{pcl}(\text{scl}(S))) = \text{scl}(cl_{\tau\alpha}(S)) = cl_{\tau\alpha}(S)$ . On the other hand we have

$$\begin{aligned} \text{pcl}(\text{scl}(\text{pcl}(S))) &= cl_{\tau\alpha}(\text{pcl}(S)) = S \cup \text{cl}(\text{int}(\text{cl}(S \cup \text{cl}(\text{int}(S))))) = \\ &= S \cup \text{cl}(\text{int}(\text{cl}(S))) = cl_{\tau\alpha}(S). \end{aligned}$$

② easily follows from ①.

## 5. WEAK FORMS OF CONTINUITY AND IRRESOLUTE-LIKE FUNCTIONS

The following irresolute-type properties have been studied in the literature: an  $f : (X, \tau) \rightarrow (Y, \sigma)$  is said to be  $\alpha$ -*irresolute* [20] (resp. *irresolute* [6], *preirresolute* [25, 35], *b-irresolute* (originally  $\gamma$ -*irresolute*) [13],  $\beta$ -*irresolute* (or *semi-preirresolute*) [23]) if  $f^{-1}(V) \in \tau^\alpha$  (resp.  $f^{-1}(V) \in \text{SO}(X, \tau)$ ,  $f^{-1}(V) \in \text{PO}(X, \tau)$ ,  $f^{-1}(V) \in \text{BO}(X, \tau)$ ,  $f^{-1}(V) \in \text{SPO}(X, \tau)$ ) for every set  $V \in \sigma^\alpha$  (resp.  $V \in \text{SO}(Y, \sigma)$ ,  $V \in \text{PO}(Y, \sigma)$ ,  $V \in \text{BO}(Y, \sigma)$ ,  $V \in \text{SPO}(Y, \sigma)$ ).

Combining [34, Theorem 3.1 and Corollary 3.1] we obtain

**Proposition 7.** Let  $(X, m_X)$  be a supratopological space and  $(Y, m_Y)$  be any  $M$ -space. Then, for a function  $f : (X, m_X) \rightarrow (Y, m_Y)$  the following statements are equivalent:

- ①  $f$  is  $M$ -continuous;
- ②  $f^{-1}(V) \in m_X$  for every  $V \in m_Y$ ;
- ③  $f(m_X - \text{cl}(S)) \subset m_Y - \text{cl}(f(S))$  for every  $S \subset X$ ;
- ④  $m_X - \text{cl}(f^{-1}(T)) \subset f^{-1}(m_Y - \text{cl}(T))$  for every  $T \subset Y$ .

Obviously, the pairs  $(X, \tau^\alpha)$ ,  $(X, \text{SO}(X, \tau))$ ,  $(X, \text{PO}(X, \tau))$ ,  $(X, \text{BO}(X, \tau))$  [3, Proposition 2.3(2)],  $(X, \text{SPO}(X, \tau))$  [2, Theorem 2.5], are supratopological spaces. Thus from Proposition 7 we can infer some characterizations of all aforementioned irresolute-type properties, for example (all other cases are left to the reader):

**Proposition 8.** *Let  $(X, \tau)$ ,  $(Y, \sigma)$  be topological spaces. The following are equivalent:*

- ①  $f : (X, \tau) \rightarrow (Y, \sigma)$  is *b-irresolute*;
- ②  $f(\text{bcl}_\tau(S)) \subset \text{bcl}_\sigma(f(S))$  for every  $S \subset X$ ;
- ③  $\text{bcl}_\tau(f^{-1}(T)) \subset f^{-1}(\text{bcl}_\sigma(T))$  for every  $T \subset Y$ .

Observe that, making use of non-trivial parts of respective results from Section 4, the above characterizations may be interestingly reformulated. For instance, using Theorem 11 we obtain

**Proposition 9.** *Let  $(X, \tau)$ ,  $(Y, \sigma)$  be topological spaces. The following are equivalent;*

- ①  $f : (X, \tau) \rightarrow (Y, \sigma)$  is *b-irresolute*;
- ②  $f(\text{bcl}_\tau(\text{spcl}_\tau(S))) \subset \text{bcl}_\sigma(\text{spcl}_\sigma(f(S)))$  for every  $S \subset X$ ;
- ③  $\text{bcl}_\tau(\text{spcl}_\tau(f^{-1}(T))) \subset f^{-1}(\text{bcl}_\sigma(\text{spcl}_\sigma(T)))$  for every  $T \subset Y$ .

Observe that also Remark 3 and Propositions 3, 3', and 6 can serve in reformulating characterizations implied by Proposition 7. For example, we have

**Proposition 10.** *For a function  $f : (X, \tau) \rightarrow (Y, \sigma)$  the following are equivalent:*

- ①  $f$  is *b-irresolute*;
- ②  $f(\text{int}(\text{scl}(S)) \cap \text{pcl}(\text{int}(S))) \subset \text{bcl}(f(S))$  for every  $S \subset X$ ;
- ③  $\text{int}(\text{scl}(f^{-1}(T))) \cap \text{pcl}(\text{int}(f^{-1}(T))) \subset f^{-1}(\text{bcl}(T))$  for every  $T \subset Y$ .

**Proof.** Easy details are omitted.

A function  $f : (X, \tau) \rightarrow (Y, \sigma)$  is said to be  $\alpha$ -continuous [31] (resp. *semi-continuous* [19], *precontinuous* [24], *b-continuous* [14],  $\beta$ -continuous [1]) if  $f^{-1}(V) \in \tau^\alpha$  (resp.  $f^{-1}(V) \in \text{SO}(X, \tau)$ ,  $f^{-1}(V) \in \text{PO}(X, \tau)$ ,  $f^{-1}(V) \in \text{BO}(X, \tau)$ ,  $f^{-1}(V) \in \text{SPO}(X, \tau)$ ) for every set  $V \in \sigma$ . Respective characterizations of all these weak forms of continuity one can easily obtain by Proposition 7. Remark 3 and Proposition 3 can also serve for this purpose – compare Proposition 10.

The following implications hold:

- (a) continuity  $\Rightarrow \alpha$ -continuity  $\Rightarrow$  semi-continuity  $\Rightarrow$  b-continuity  $\Rightarrow \beta$ -continuity;
- (b)  $\alpha$ -continuity  $\Rightarrow$  precontinuity  $\Rightarrow$  b-continuity;
- (c)
  - $\alpha$ -irresoluteness  $\Rightarrow \alpha$ -continuity;
  - irresoluteness  $\Rightarrow$  semi-continuity;
  - preirresoluteness  $\Rightarrow$  precontinuity;

④ b-irresoluteness  $\Rightarrow$  b-continuity;

⑤  $\beta$ -irresoluteness  $\Rightarrow$   $\beta$ -continuity.

Recall that semi-continuity and precontinuity are independent of each other [27] (it should be remarked that precontinuity coincides with almost continuity in the sense of Husain [24]).

**Lemma 7.** *A function  $f : (X, \tau) \rightarrow (Y, \sigma)$  is  $\beta$ -continuous if and only if  $f(\text{spcl}(S)) \subset \text{cl}(f(S))$  for any  $S \subset X$ .*

**Proof.** Put in Proposition 7,  $m_X = \text{SPO}(X, \tau)$  and  $m_Y = \sigma$ .

**Corollary 5.** *Let  $f : (X, \tau) \rightarrow (Y, \sigma)$  be  $\beta$ -continuous. Then  $\text{cl}(f(\text{spcl}(S))) = \text{cl}(f(S))$  for any  $S \subset X$ .*

**Proof.** Using Lemma 7 we obtain.

$$\text{cl}(f(S)) \subset \text{cl}(f(\text{spcl}(S))) \subset \text{cl}(f(S)).$$

A function  $f : (X, \tau) \rightarrow (Y, \sigma)$  is said to be **a.o.W** [38] if  $f^{-1}(\text{cl}(V)) \subset \text{cl}(f^{-1}(V))$  for every  $V \in \sigma$ .  $f$  is said to be **a.o.S.** [37] if  $f(U) \in \sigma$  for each  $U \in \text{RO}(X, \tau)$ .  $f$  is said to be **semi-open** [4] if  $f(U) \in \text{SO}(Y, \sigma)$  for each  $U \in \tau$ .  $f$  is said to be **weakly open** [36] if  $f(U) \subset \text{int}(f(\text{cl}(U)))$  for each  $U \in \tau$ .

For the above openness-like notions, Noiri [30, p.315] showed that ① a.o.S. and a.o.W., ② a.o.S. and semi-openness, ③ a.o.W. and semi-openness, ④ a.o.W. and weak openness, ⑤ semi-openness and weak openness, are all the five couples of independent notions. Noiri has also shown [30, Lemma 1.4] that a.o.S. implies weak openness, while the converse is false in general [30, Example 1.5].

**Definition 2.** Let an  $f : (X, \tau) \rightarrow (Y, \sigma)$ .

- [10, Definition 3.27]  $f$  is said to be **contra-semiopen** if  $f(U) \in \text{SC}(Y, \sigma)$  for each  $U \in \tau$ .
- $f$  is said to be **R-open** if  $f(U) \in \text{RO}(Y, \sigma)$  for each  $U \in \text{RO}(X, \tau)$ .

Corollary 5 is very useful to investigate the set-structure of the family of all R-open functions. In what follows, e.d. stands for extremal disconnectedness condition:  $\text{cl}(S) \in \tau$  for all  $S \in \tau$ .

**Theorem 13.** *Assume  $f : (X, \tau) \rightarrow (Y, \sigma)$  is contra-semiopen.*

- ① *If  $f$  is  $\beta$ -continuous and a.o.W., then it is R-open.*
- ② *If  $f$  is semi-continuous and semi-open, then it is R-open.*
- ③ *If  $(Y, \sigma)$  is e.d.,  $f$  is  $\beta$ -continuous and semi-open, then  $f$  is R-open.*
- ④ *If  $f$  is pre-continuous and weakly open, then it is R-open.*

**Proof.** We can easily check that  $\text{spcl}(U) = \text{int}(\text{cl}(U))$  for any open  $U$ . So, by Corollary 5 we get  $\text{int}(\text{cl}(f(U))) = \text{int}(\text{cl}(f(\text{int}(\text{cl}(U))))$ . Then, making use of respectively [10, Lemmas 3.28, 3.30, 3.32, 3.34] we obtain in all cases ①–④  $f(U) = \text{int}(\text{cl}(f(U)))$  for any  $U \in \text{RO}(X, \tau)$ .

In [6, Theorem 1.2] it has been shown that each continuous and open function is irresolute. A stronger result we obtain below.

**Theorem 14.** *Let a function  $f : (X, \tau) \rightarrow (Y, \sigma)$  be continuous and open. Then  $f$  is:*  
 ①  $\alpha$ -irresolute, ② irresolute, ③ preirresolute, ④  $b$ -irresolute, ⑤  $\beta$ -irresolute.

**Proof.** We shall only ⑤ (proofs for other cases are similar). Let  $S \subset X$  be arbitrary. As  $f$  is continuous and open, we get the inclusions

$$f(\text{cl}(\text{int}(S))) \subset \text{cl}(\text{int}(f(S))) \text{ and } f(\text{int}(\text{cl}(S))) \subset \text{int}(\text{cl}(f(S))).$$

Then

$$f(\text{cl}(\text{int}(S)) \cap \text{int}(\text{cl}(S))) \subset \text{cl}(\text{int}(f(S))) \cap \text{int}(\text{cl}(f(S))),$$

and by the formula for  $\text{bcl}(S)$  (Proposition 1 ④) we obtain

$$f(\text{bcl}(S)) \subset \text{bcl}(f(S)).$$

Therefore, by Proposition 8,  $f$  is  $b$ -irresolute.

**Corollary 6.** *Among continuous open mappings all irresoluteness-like notions listed in Theorem 14 are equivalent.*

**Corollary 7.** *Let an  $f : (X, \tau) \rightarrow (Y, \sigma)$  be continuous, open, and contra-semiopen. Then it is  $R$ -open.*

**Proof.** Simply apply Theorems 14 and 13. Let us remark on the occasion that since  $\text{RO}(Y, \sigma) = \sigma \cap \text{SC}(Y, \sigma)$  [7, Lemma 2.2], we have for any open and contra-semiopen function  $f$  that  $f(U) \in \text{RO}(Y, \sigma)$  for each  $U \in \tau$ .

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# NON-PLANAR GRAPHS FROM VEP GRAPHS CLASS-1

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**ABSTRACT :** In this paper we construct a set of vertex extension planar graphs (VEP-graphs) and from VEP graph  $G_e(2m + 2, 6m)$ ,  $m \geq 2$ , we construct a set of non-planar graphs  $G_{NP-1}$ . In this set of  $G_{NP-1}$  we have done some theoretical investigations relating to isomorphism, maximal clique, edge disjoint Hamiltonian circuit, minimum and maximum number of crossing and an application.

AMS Subject classification: 05c30, 05c45.

## 1.1 INTRODUCTION

Kalita [1] introduced the concept of TEEP-graphs and many properties of TEEP-graphs are discussed by him. The TEEP-graphs have been used for the construction of strongly regular graphs [2]. A new class of non-planar graphs [3] are constructed from TEEP-graphs with certain properties relating to maximum and minimum number of crossings. It is found that certain types of TEEP-graphs have nice application for the solution of traveling salesman problems [4, 5]. Recently Kalita & Sarma [6] constructed VEP-graphs (vertex extension planar-graphs)  $G_0(2m + 3, 6m + 3)$  and  $G_e(2m + 2, 6m)$  for  $m \geq 2$  with odd and even number of vertices respectively. From the VEP-graphs  $G_0(2m + 3, 6m + 3)$  for  $m \geq 2$  they constructed a set of non-planar graphs  $G_{NP} = \{G_i(2m + 4, 8m + 6), m \geq 2, i = 1, 2, \dots, (4m + 1)\}$  and discussed different properties of the set of graphs  $G_{NP}$ .

In this paper we construct a set of non-planar graphs  $G_{NP-1}$  from the VEP-graph  $G_e(2m + 2, 6m)$  for  $m \geq 2$  and study some properties of them with an application of these non-planar graphs.

Before going to construct the non-planar graphs from the VEP-graph  $G_e(2m + 2, 6m)$  for  $m \geq 2$  let us define the VEP-graph and remind the construction process and properties of VEP-graphs.

### 1.1 (a) DEFINITION OF VEP (vertex extension planar) GRAPHS:

A planar graph constructed from the complete graph  $K_3$  by extending a vertex each time in the outer region and joining this vertex to the three vertices in the outer region by three edges is called a VEP (vertex extension planar) graphs.

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## 1.2 CONSTRUCTION OF VERTEX EXTENSION PLANAR GRAPHS: (VEP-graphs)

We consider the complete graph  $K_3$ .  $K_3$  has two regions. One is interior and other is exterior region. [Generally the complete graphs has no regions, but the representation of  $K_3$  can be considered as planar with two regions as mentioned]. In the exterior region of  $K_3$  we take a vertex say  $P_1$  and join this vertex with the vertices of  $K_3$  by edges and get a planar graph with even number of vertices and we denote this graph as  $G_e$  (Fig:1)

[The suffix e is used to specify that the graph has even number of vertices]

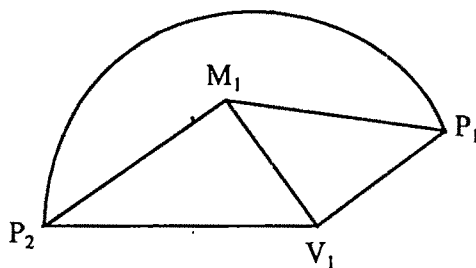


Fig: 1

In this graph, we have four vertices. One of them is in the interior regions of  $G_e$ , we may call this vertex as  $M_1$  and remaining three vertices are in the exterior region of  $G_e$ . One is the vertex  $P_1$  and the other two we may label them as  $P_2$  and  $V_1$ . Now in the exterior region of  $G_e$  we take a new vertex  $V_2$  and connect  $V_2$  to the vertices  $P_1$ ,  $V_1$  and  $P_2$  of the graph  $G_e$  by edges and get a planar graph  $G_o$  of odd number of vertices (i.e. 5) (Fig: 2) [here o is used to specify odd]

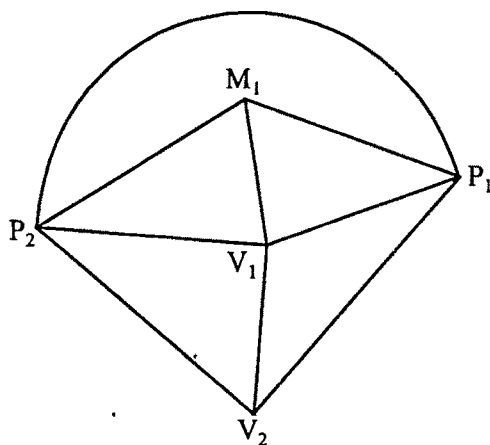


Fig: 2



Again in the exterior region of  $G_0$  we take a new vertex  $V_3$  and connect  $V_3$  to the vertices  $V_2$ ,  $P_1$  and  $P_2$  of the graph  $G_0$  by edges and get a planar graph  $G_e$  (Fig: 3).

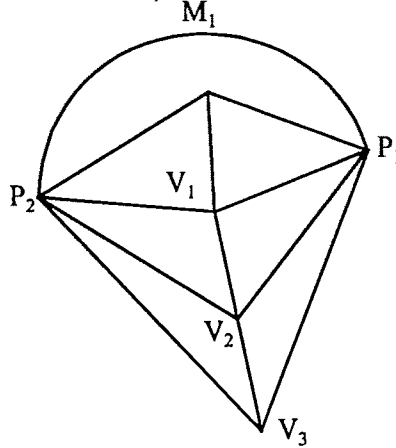


Fig: 3

In the exterior region of this  $G_e$  we take a new vertex  $V_4$  and connect to the vertices  $V_3$ ,  $P_1$  and  $P_2$  of the graph  $G_e$  by edges and get a planar graph  $G_0$  of odd number of vertices. Continuing this process we get two planar graphs  $G_e(2m + 2, 6m)$  and  $G_0(2m + 3, 6m + 3)$ ,  $m \geq 2$  having  $4m$  and  $(4m + 2)$ ,  $m \geq 2$  number of regions respectively.

We have obtained these sets of planar graphs by extending vertices i.e. first we extend vertex from 3 to 4, then 4 to 5, then 5 to 6 and so on. So we call these sets as vertex extension planar graphs: (VEP-graphs)  $G_e(2m + 2, 6m)$  and  $G_0(2m + 3, 6m + 3)$ ,  $m \geq 2$

### 1.3 PROPERTIES OF VEP-GRAPHS

In these VEP-graphs  $G_e(2m + 2, 6m)$  and  $G_0(2m + 3, 6m + 3)$ ,  $m \geq 2$  we have

- (1) Two vertices of minimum degree 3 i.e.  $\delta(G_0) = 3$  and  $\delta(G_e) = 3$  [The minimum degree of a graph is denoted by  $\delta$ ]
- (2) Two vertices of maximum degree  $(2m + 2)$  of the graph  $G_0(2m + 3, 6m + 3)$ ,  $m \geq 2$  i.e.  $\Delta(G_0) = (2m + 2)$  [The maximum degree of a graph is denoted by  $\Delta$ ]
- (3) Two vertices of maximum degree  $(2m + 1)$  of the graph  $G_e(2m + 2, 6m)$  for  $m \geq 2$  i.e.  $\Delta(G_e) = (2m + 1)$
- (4) Other vertices are of degree 4 for both the graphs  $G_e(2m + 2, 6m)$  and  $G_0(2m + 3, 6m + 3)$ ,  $m \geq 2$

Note: This VEP graph is different from TEEP graphs as the properties 2, 3, 4 of VEP graphs are different from the properties of TEEP graphs.

### 1.4 NOTATION AND TERMINOLOGY

In the VEP graphs we have two vertices of maximum degree. These two vertices are  $P_1$  and  $P_2$ . Two vertices of minimum degree one is  $M_1$  and we denote the other as  $M_2$ . Also we have  $(2m - 2)$  number of vertices of degree 4. We denote these vertices as  $V_1, V_2, V_3 \dots V_{2m-2}$ . Every interior region of VEP-graphs is covered by three edges joining three vertices. But for our convenience we say that three vertices cover every interior region of VEP-graphs. Now in the graph  $G_e(2m + 2, 6m)$ ,  $m \geq 2$  out of  $(4m - 1)$  interior regions we have five interior regions, at least one vertex of each region is of minimum degree, one vertex is of maximum degree. These five regions together we will call as Principal Regions (PR-regions). Other regions of this graph i.e.  $(4m - 6)$  interior regions are covered by two vertices of degree 4 and one is either  $P_1$  or  $P_2$ .

Now we construct a set of non-planar graphs from the VEP-graph  $G_e(2m + 2, 6m)$ ,  $m \geq 2$ . We shall denote this set of non-planar graphs by  $G_{NP-1}$  and each member of this set by  $G_i(2m + 3, 8m + 2)$  for  $m \geq 2$ .

### 2.1 CONSTRUCTION OF NON-PLANAR GRAPHS FROM VEP-GRAPH

$$G_e(2m + 2, 6m), m \geq 2$$

Let us consider the VEP-graph  $G_e(2m + 2, 6m)$ ,  $m \geq 2$ . For  $m = 2$ , the graph has seven interior regions and one exterior region. We introduce a new vertex  $V$  in any interior region of this graph. Connecting this vertex to the other vertices by edges, we obtain a new graph which is non-planar graph (Fig-4), as some edges of this new graph intersect at some edges of previous graph. Thus taking the vertex  $V$  in any one of the seven interior regions and connecting to the other vertices we get such seven non-planar graphs.

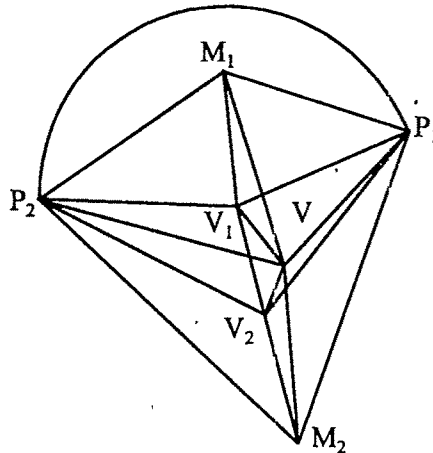


Fig-4

Similarly in the VEP graph  $G_e(2m + 2, 6m)$ ,  $m \geq 2$ , we have  $(4m - 1)$  interior regions and taking a new vertex  $V$  in any of these  $(4m - 1)$  interior regions and joining this vertex by edges to the other vertices we get a set of non-planar graphs  $G_{NP-1}$ . This set has  $(4m - 1)$  number of elements (non-planar graphs)

Thus  $G_{NP-1} = \{G_i(2m + 3, 8m + 2), m \geq 2, i = 1, 2, \dots, (4m - 1)\}$ . We study the isomorphism property, maximum clique edge disjointed Hamiltonian circuits, minimum number of crossing and maximum number of crossing for the members of the set  $G_{NP-1}$  and we have developed the following theorems.

Note: The set  $G_{NP-1} = \{G_i(2m + 3, 8m + 2), m \geq 2, i = 1, 2, \dots, (4m - 1)\}$  is non planar as each member of the set of graphs some edges of this new graph intersect at some edges of the previous graph (one can not draw  $G_{NP-1}$  without intersecting some edges of the previous graph)

**Theorem 1.** All members (graph) of the set  $G_{NP-1}$  are isomorphic to each other.

**Proof.** Let us consider any two members of the set  $G_{NP-1}$ . Let  $G_i$  and  $G_j$  be any two members of the set  $G_{NP-1}$ . Now from the construction process of the members of the set  $G_{NP-1}$  we see that pattern of each member are same. The minimum degree of each member is 4 and the maximum degree is  $(2m + 2)$  for  $m \geq 2$  and degrees of the other vertices are 5. Hence there exists a one-one correspondence between the vertices of  $G_i$  and  $G_j$  which preserves the adjacency property of isomorphic graphs. Hence  $G_i$  and  $G_j$  are isomorphic to each other. So all members of the set  $G_{NP-1}$  are isomorphic to each other which proves the theorem.

**Theorem 2.** Each member (graph) of the set  $G_{NP-1}$  has a maximum clique graph  $K_5$ .

**Proof.** From the construction process of the planar graph  $G_e(2m + 2, 6m)$ ,  $m \geq 2$ , we see that the vertices of the complete graph  $K_4$  are  $P_1, P_2, M_1$  and  $V_1$ . Taking a vertex  $V$  in any interior region of the planar graph  $G_e(2m + 2, 6m)$ ,  $m \geq 2$  and joining this vertex to the other vertices of the planar graph  $G_e(2m + 2, 6m)$ ,  $m \geq 2$  by edges we get the set of non planar graph  $G_{NP-1}$ . The new vertex  $V$  is connected to all four vertices  $P_1, P_2, M_1$  and  $V_1$  and these five vertices form the complete graph  $K_5$ . No other vertices i.e.  $V_2, V_3, \dots, V_{2m-2}$  are not connected to the vertex  $M_1$ . Similarly  $V_3$  is also not connected to  $V_1, V_4$  is also not connected to  $V_1$  and  $V_2$  and so on. Thus  $K_5$  is the maximum clique graph of the set  $G_{NP-1}$  e.g. for  $m = 2$  (Fig.-5) the vertex  $V_2$  is not connected to the vertex  $M_1$  by edges. Similarly the vertex  $M_2$  is also not connected to the vertices  $M_1$  and  $V_1$ . Only the complete graph  $K_5$  whose vertices are  $P_1, P_2, M_1, V_1$  and  $V$  is the maximum clique graph for  $m = 2$ . Hence prove.

**Theorem 3.** The maximum number of crossing exists in the graph  $G_i(2m + 3, 8m + 2)$ ,  $m \geq 2$  of the set  $G_{NP-1}$  if the new vertex  $V$  is introduced in any one of the PR-region of the graph  $G_e(2m + 2, 6m)$ ,  $m \geq 2$  and the maximum number of crossing is  $m^2$  for  $m \geq 2$ .

**Proof.** We know that there are five interior regions in the PR-region. Three of them are the interior regions of the complete graph  $K_4$  and they are adjacent to each other. Each of those three regions is a furthest region from the other regions. When we introduce the new vertex  $V$  in one of these three regions and connect the vertex  $V$  with other vertices of the graph  $G_e(2m+2, 6m)$ ,  $m \geq 2$  then automatically the crossing of all edges will be maximum. Similarly the other two interior region of the PR-region is obtained by joining the minimum degree vertex  $M_2$  to  $P_1$  and  $P_2$  by edges and these two interior regions are also adjacent to each other and furthest from the others. If we introduce the new vertex  $V$  in these regions and connect this vertex with all other vertices of the graph  $G_e(2m+2, 6m)$ ,  $m \geq 2$  then automatically the crossing of all edges will be maximum. The second part of the theorem i.e. maximum number of crossing can be easily proved by mathematical induction.

**Theorem 4.** The minimum number of crossing exists in two regions of the graph  $G_{NP-1}$ , covered by  $V_m, V_{m-1}$  and  $P_1$  vertices and the region covered by  $V_m, V_{m-1}$  and  $P_2$  vertices and the number is  $m(m-1)+1$ ,  $m \geq 2$ .

**Proof.** In the graph of  $G_e(2m+2, 6m)$ ,  $m \geq 2$  we have  $2(m-1)$  i.e. even number of vertices of degree 4. So the interior regions covered by  $V_m, V_{m-1}$  and  $P_1$  vertices and the region covered by  $V_m, V_{m-1}$  and  $P_2$  vertices will be exists in the middle of the graph. So if we introduce the new vertex  $V$  in these regions and connect  $V$  to other vertices by edges we have to cross minimum number of edges and hence the number of crossing is minimum which proves the first part of the theorem.

The second part of the theorem can be proved by mathematical induction.

**Theorem 5.** Every member of the set  $G_{NP-1}$  is Hamiltonian and for  $m = 2$  every member of the set  $G_{NP-1}$  have two edge disjoint Hamiltonian circuits but for  $m \geq 3$  there does not exist any edge disjoint Hamiltonian circuits.

**Proof.** Let  $G_e(2m+3, 8m+2)$  for  $m \geq 2$  be any member of the set  $G_{NP-1}$ . The vertices of the graph  $G_i$  are  $P_1, P_2, M_1, V_1, V_2, V_3, \dots, V_{2m-2}, M_2$ , and  $V$ . We consider any vertex, say  $V_4$  then from construction process we have the graphs  $V_4V_5V_6V_7 \dots V_{2m-2}M_2P_2M_1VV_1V_2V_3V_4$  is Hamiltonian [A circuit connecting all vertices]. From Fig-5 for  $m = 2$  we see that there are two edge disjoint Hamiltonian circuits.

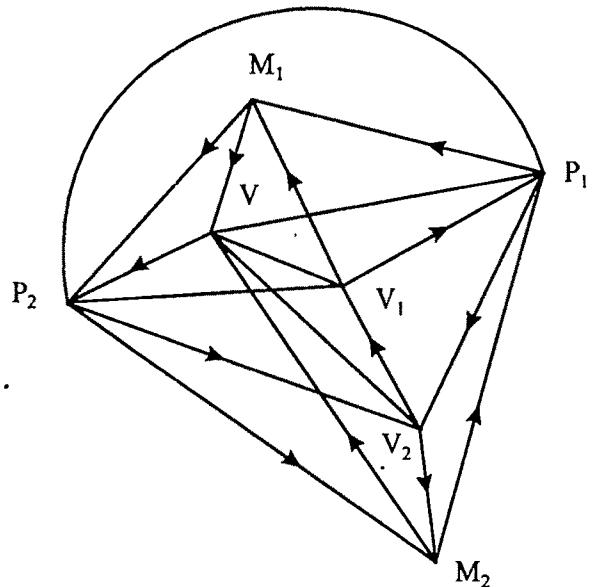


Fig. 5

But for  $m \geq 3$ , if we can show that at least one member of the set  $G_{NP-1}$  has no edge disjoint Hamiltonian circuit then the second part will be proved. For this we draw a graph for  $m = 3$  and from the graph (Fig-6) it is clear that there does not exist any disjoint Hamiltonian circuit.

**Theorem 6.** Every member of the set  $G_{NP-1} = \{G_i(2m + 3, 8m + 2), m \geq 2, i = 1, 2, \dots (4m - 1)\}$  is non planar.

**Proof.** We know that a graph will be non planar [7] if  $e > (3n - 6)$ , where  $e$  is the number of edges and  $n$  is the number vertices. Here for each member of the graph  $G_{NP-1} = G_i(2m + 3, 8m + 2, m \geq 2, i = 1, 2, \dots (4m - 1))$ , the number of edges  $e = 8m + 2$ , and the number of vertices  $n = 2m + 3$  for  $m \geq 2$ . Now  $3n - 6 = 3(2m + 3) - 6 = 6m + 3$  for  $m \geq 2$  and  $e = 8m + 2 = 6m + 3 + 2m - 1 = (3n - 6) + 2m - 1 > (3n - 6)$ , for  $m \geq 2$  Thus the condition  $e > (3n - 6)$ , is satisfied. Which proves the theorem.

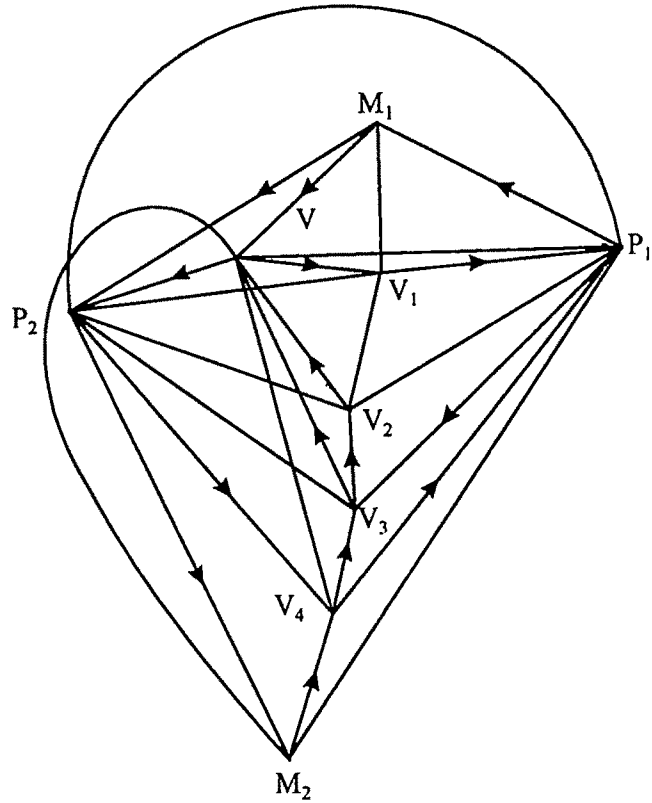


Fig. 6

### APPLICATION OF RESEARCH FINDINGS

**Problem.** Suppose there are seven main police points in a city and there are eighteen roads connecting all these police points. If the traffic density is more due to the increasing of population in the city, then definitely, in every crossing of two roads, there must have traffic police duty in every crossing of roads.

Now design a network with the given eighteen roads and seven police stations so that there should have minimum traffic police duty at every crossing of roads.

**Solution.** The network can be designed from the non-planer graph  $G_i(2m + 3, 8m + 2)$  for  $m = 2$ , which satisfy all the conditions of the given problem.

The minimum number of police duty i.e. minimum number of crossing can be obtained

from the theorem: 4 which is given  $m(m-1) + 1 = 2.1 + 1 = 3$  (for  $m = 2$ ) so the minimum number of traffic police duty at these crossing points is 3.

Similarly one can design a network for different values of  $m$  by using the non-planar graph  $G_1(2m+3, 8m+2)$ ,  $m > 2$  and the minimum number of police duty at the crossing can be obtained by using the theorem: 4.

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## SOMEWHAT CONTINUOUS FUNCTIONS ON FUZZIFIED TOPOLOGICAL SPACE

TAZID ALI AND SAMPA DAS

**ABSTRACT :** In this paper the concept of somewhat continuous functions, somewhat open functions are introduced in fuzzified topological space and some interesting properties of these functions are investigated.

**Key words :** Fuzzified topological space, somewhat continuous function, somewhat open function.  
1991 *Mathematics Subject Classification.* 54C08.

### 1. INTRODUCTION

The theory of fuzzy topological spaces was introduced and developed by C.L. Chang [2] and since then various notions in classical topology have been extended to fuzzy topological spaces. The concept of somewhat functions was introduced by Karl R. Genry and Hughes B. Hoyle [3] and this concept was studied in connection with the idea of feebly continuous function and feebly open functions introduced by Zdenek Frolik [4]. In [5] G. Thangaraj and G. Balasubramanian introduced these concepts in fuzzy topological spaces. In this paper we have discussed the properties of somewhat continuous and somewhat open functions in fuzzified topological spaces.

### 2. PRELIMINARIES

In this section we recall some definitions and results that will be used in the sequel.

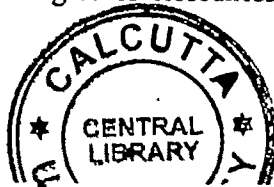
**Definition 2.1**[7]. A fuzzified topology on a nonempty set  $X$  is fuzzy subset of power set of  $X$ ,  $P(X)$  i.e., a function  $\tau : P(X) \rightarrow [0, 1]$ , satisfying the following conditions:

$$\tau(X) = \tau(\phi) = 1.$$

$$\tau(A \cap B) \geq \tau(A) \wedge \tau(B), A, B \in P(X)$$

$$\tau(\cup A_i) \geq \bigwedge_i \tau(A_i) \text{ for any sub collection } \{A_i\} \text{ of } P(X).$$

The pair  $(X, \tau)$  will be called a fuzzified topological space (FTS).  $\tau(A)$  is the degree of openness of  $A$  and  $\tau(A^c)$  is the degree of closedness of  $A$ , where  $A^c$  is the complement of  $A$ .



**Definition 2.2.**[7] A fuzzified co-topology on a nonempty set  $X$  is a fuzzy subset of  $P(X)$  i.e., a function  $\omega : P(X) \rightarrow [0, 1]$ , satisfying the following conditions:

$$\omega(X) = \omega(\phi) = 1$$

$$\omega(A \cup B) \geq \omega(A) \wedge \omega(B), A, B \in P(X)$$

$$\tau(\cap A_i) \hat{=} \tau(A_i) \text{ for any sub collection } \{A_i\} \text{ of } P(X).$$

**Proposition 2.3**[7] Let  $\tau$  be a fuzzified topology on  $X$  and  $\omega_\tau : X \rightarrow [0, 1]$  be defined as  $\omega_\tau(A) = \tau(A^c)$ . Then  $\omega_\tau$  is a fuzzified co-topology on  $X$ .

**Proposition 2.4**[7] Let  $\omega$  be a fuzzified co-topology on  $X$  and  $\tau_\omega : X \rightarrow [0, 1]$  be defined as  $\tau_\omega(A) = \omega(A^c)$ . Then  $\tau_\omega$  is a fuzzified topology on  $X$ .

**Proposition 2.5**[7] If  $\tau$  is a fuzzified topology and  $\omega$  is a fuzzified co-topology on  $X$  then  $\tau_{\omega\tau} = \tau$  and  $\omega_{\tau\omega} = \omega$ .

**Proposition 2.6**[7] Let  $(X, \tau)$  be a fuzzified topology and  $Y \subseteq X$ . Then  $\tau_Y : P(Y) \rightarrow [0, 1]$  given by

$$\tau_Y(U) = \vee \{\tau(V) : U = Y \cap V\} \text{ is a fuzzified topology on } Y.$$

$\tau_Y$  is then called fuzzified subspace topology.

**Definition 2.7**[7] Let  $(X, \tau)$  and  $(Y, \delta)$  be two FTSSs. A function  $f : (X, \tau) \rightarrow (Y, \delta)$  is said to be continuous with respect to  $\tau$  and  $\delta$  if  $\tau(f^{-1}(U)) \geq \delta(U)$  for each  $U \in P(Y)$ .

**Result 2.8** Let  $(X, \tau)$  and  $(Y, \delta)$  be two FTSSs. A function  $f : (X, \tau) \rightarrow (Y, \delta)$  is continuous with respect to  $\tau$  and  $\delta$  iff  $\omega_\tau(f^{-1}(U)) \geq \omega_\delta(U)$  for each  $U \subseteq Y$ .

**Definition 2.9**[7] Let  $(X, \tau)$  and  $(Y, \delta)$  be two FTSSs. A function  $f : (X, \tau) \rightarrow (Y, \delta)$  is said to be open with respect to  $\tau$  and  $\delta$  if  $\tau(G) \leq \delta(f(G))$  for each  $G \in P(X)$ .

**Definition 2.10**[7] Let  $(X, \tau)$  and  $(Y, \delta)$  be two FTSSs. A function  $f : (X, \tau) \rightarrow (Y, \delta)$  is said to be closed with respect to  $\tau$  and  $\delta$  if  $\omega_\tau(G) \leq \omega_\delta(f(G))$  for each  $G \in P(X)$ .

### 3. SOMEWHAT CONTINUOUS FUNCTIONS IN FUZZIFIED TOPOLOGICAL SPACE

In this section we introduce and discuss the properties of somewhat continuous function in fuzzified topological spaces.

**Definition 3.1** Let  $(X, \tau)$  and  $(Y, \delta)$  be two FTSSs. A function  $f : (X, \tau) \rightarrow (Y, \delta)$  is said to be somewhat continuous with respect to  $\tau$  and  $\delta$  if for each  $U \subseteq Y$ , with  $f^{-1}(U) \neq \phi$  there exist  $\phi \neq V \subseteq X$  such that  $V \subseteq f^{-1}(U)$  and  $\tau(V) \geq \delta(U)$ .



It is clear from the definition that a continuous function is somewhat continuous. That the converse is not always true is evident from the following example.

**Example 3.2** Let  $X = \{a, b, c\}$ ,  $\mu$  and  $\nu$  are fuzzified topologies on  $X$  defined as :

	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	$\phi$	$X$
$\mu$	.7	.5	.1	.5	.3	.1	1	1
$\nu$	.1	0	.1	.6	.1	0	1	1

Let  $g : (X, \mu) \rightarrow (Y, \nu)$  be the identity function. Then  $\mu[g^{-1}(\{a, b\})] = \mu(\{a, b\}) = .5 < .6 = \nu(\{a, b\})$ . Therefore  $g$  is not continuous. However there exists  $\{a\} \subseteq \{a, b\}$  such that  $\mu(\{a\}) > \nu(\{a, b\})$ . Hence  $g$  is somewhat continuous.

**Result 3.3.** A function  $f : (X, \tau) \rightarrow (Y, \delta)$  is somewhat continuous with respect to  $\tau$  and  $\delta$  iff for each  $U \subseteq Y$ , with  $f^{-1}(U) \neq X$  there exist  $V \subsetneq X$  such that  $f^{-1}(U) \subseteq V$  and  $\omega_\tau(V) \geq \omega_\delta(U)$ .

**Proof.** Let  $f : (X, \tau) \rightarrow (Y, \delta)$  be somewhat continuous. Consider  $U \subseteq Y$  such that  $f^{-1}(U) \neq X$ . Then  $U^c \subseteq Y$  such that  $f^{-1}(U^c) \neq \phi$ .

As  $f$  is somewhat continuous, there exists  $\phi \neq W \subseteq X$  :

$$W \subseteq f^{-1}(U^c) \text{ and } \tau(W) \geq \delta(U^c) \Rightarrow W \subseteq \{f^{-1}(U)\}^c \text{ and } \omega_\tau(W^c) \geq \omega_\delta(U)$$

$$\Rightarrow f^{-1}(U) \subseteq W^c \text{ and } \omega_\tau(W^c) \geq \omega_\delta(U).$$

$$\text{Let } V = W^c, \text{ then } V \subsetneq X : f^{-1}(U) \subseteq V \text{ and } \omega_\tau(V) \geq \omega_\delta(U)$$

Converse.

Let  $U \subseteq Y$  such that  $f^{-1}(U) \subsetneq X$ . Then  $U^c \subseteq Y$  such that  $f^{-1}(U^c) \neq \phi$ . So by hypothesis, there exist  $V \subsetneq X : f^{-1}(U^c) \subseteq V$  and  $\omega_\tau(V) \geq \omega_\delta(U^c)$  which gives  $\{f^{-1}(U)\}^c \subseteq V$  and  $\omega_\tau(V) \geq \omega_\delta(U^c) \Rightarrow V^c \subseteq f^{-1}(U)$  and  $\tau(V^c) \geq \delta(U)$ . Let  $V^c = W$ .

$$\text{Then } W \neq \phi \text{ and } W \subseteq f^{-1}(U) \text{ and } \tau(W) \geq \delta(U)$$

**Result 3.4** Let  $(X, \tau)$ ,  $(Y, \delta)$  and  $(Z, \eta)$  be FTSSs.

Let  $f : (X, \tau) \rightarrow (Y, \delta)$  and  $g : (Y, \delta) \rightarrow (Z, \eta)$  are somewhat continuous functions, then  $g \circ f : (X, \tau) \rightarrow (Z, \eta)$  is continuous, if  $f^{-1}(V) \neq \phi$  for all  $\phi \neq V \subseteq Y$ .

**Proof.** Let  $U \subseteq Z$  such the  $(g \circ f)^{-1}(U) \neq \phi$ .

We have  $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U) \neq \phi$  implies  $g^{-1}(U) \neq \phi$ . As  $g$  is somewhat continuous, there exists  $\phi \neq W \subseteq Y : W \subseteq g^{-1}(U)$  and  $\delta(W) \geq \eta(U)$ .

Again by hypothesis  $\phi \neq W \Rightarrow f^1(W) \neq \phi$ . As  $f$  is somewhat continuous, there exists  $\phi \neq V \subseteq X : V \subseteq f^1(W)$  and  $\tau(V) \geq \delta(W)$ . Thus  $V \subseteq f^1(g^{-1}(U))$  and  $\tau(V) \geq \delta(W) \geq \eta(U)$ .

That is, there exists  $\phi \neq V \subseteq X : V \subseteq (gof)^{-1}(U)$  and  $\tau(V) \geq \eta(U)$ .

The following example shows that the condition  $f^1(V) \neq \phi$  for all  $\phi \neq V \subseteq Y$  is sufficient but not necessary.

**Example 3.5** Let  $X = \{a, b, c\}$ ,  $\tau$ ,  $\mu$  and  $\nu$  are fuzzified topologies on  $X$  defined as:

	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	$\phi$	$X$
$\mu$	.7	.5	.1	.5	.3	.1	1	1
$\nu$	.1	0	.0	.6	.1	0	1	1
$\tau$	.7	.4	0	.4	0	.1	1	1

Let  $f : (X, \mu) \rightarrow (X, \tau)$  be defined as  $f(a) = a$ ,  $f(b) = f(c) = c$  and  $g : (X, \tau) \rightarrow (X, \nu)$  be the identity function.

Then it is easy to verify that  $f$ ,  $g$  and  $gof$  are somewhat continuous though  $f^1(\{b\}) = \phi$ .

**Definition 3.6** Let  $(X, \tau)$  be a FTS. A subset  $D \subseteq X$  is said to be dense in  $X$  if there exists no subset  $V \subsetneq X$  such that  $D \subseteq V$  and  $\omega_\tau(V) > \omega_\tau(D)$ .

**Result 3.7** Image of a dense subset under a somewhat continuous and closed function is dense.

**Proof.**  $f : (X, \tau) \rightarrow (Y, \mu)$  be somewhat continuous and  $D$  be dense in  $X$ . Consider  $f(D)$ . If  $f(D) = Y$ , we are done. So let  $f(D) \neq Y$ . Suppose  $f(D)$  is not dense, then there exists  $B \subsetneq Y : f(D) \subsetneq B$  and  $\omega_\mu(B) > \omega_\mu(f(D))$ . Since  $f$  is somewhat continuous, for  $B \subsetneq Y$  there exists  $F \subsetneq X : f^1(B) \subseteq F$  and  $\omega_\tau(F) \geq \omega_\mu(B)$ . Again  $f(D) \subseteq B \Rightarrow D \subseteq f^1(B) \subseteq F \Rightarrow D \subseteq F$ . Also,  $\omega_\tau(F) \geq \omega_\mu(B) > \omega_\mu(f(D)) \geq \omega_\tau(D)$ , since  $f$  is closed. But this is a contradiction to the fact that  $D$  is dense in  $X$ .

The following example shows that the above result need not be true if the condition of closed of  $f$  is omitted.

**Example 3.8** Let  $X = \{a, b, c, d\}$  and  $\tau$  and  $\mu$  be fuzzified topologies on  $X$  defined as:

$$\begin{aligned} \tau(A) &= 1, A = X, \phi \\ &= .7, A = \{c, d\} \\ &= .5, \text{ otherwise.} \end{aligned}$$

$$\begin{aligned}\mu(A) &= 1, A = X, \phi \\ &= .6, A = \{b\} \\ &= .4, \text{ otherwise.}\end{aligned}$$

Let  $f : (X, \tau) \rightarrow (Y, \mu)$  be given  $f(a) = b, f(b) = c, f(c) = d, f(d) = a$ .

Then  $f$  is somewhat continuous and  $D = \{a, b\}$  is dense in  $X$ .

But  $f(D) = \{b, c\}$  is not dense in  $Y$ , as there exists  $\{a, c, d\} \supsetneq \{b, c\}$  such that

$$\omega_\mu(\{a, c, d\}) = \mu(\{b\}) = .6 > .4 = \mu(\{a, d\}) = \omega_\mu(\{b, c\}) = \omega_\mu(f(D)).$$

Note that here  $f$  is not closed.

**Result 3.9** Let  $f : (X, \tau) \rightarrow (Y, \delta)$  be somewhat continuous. Then for any  $A \subseteq X$ ,

$f|_A : (A, \tau_A) \rightarrow (Y, \delta)$  is somewhat continuous.

**Proof.** Since  $f|_A$  is the composition of the inclusion map  $j : (A, \tau_A) \rightarrow (X, \tau)$  and  $f : (X, \tau) \rightarrow (Y, \delta)$ , the result follows from Result 3.4.

**Result 3.10** Let  $f : (X, \tau) \rightarrow (Y, \delta)$  be somewhat continuous where  $Y$  is a subspace of  $Z$ . Then  $h : X \rightarrow Z$ , where  $h$  is obtained by expanding the range of  $f$  is somewhat continuous.

**Proof.** Let  $\mu$  denote the fuzzified topology on  $Z$ . Then  $\delta = \mu_Y$ . Let  $U \subseteq Z$  with  $h^{-1}(U) \neq \phi$ . We have  $V = Y \cap U \subseteq U$ . Then  $f^{-1}(V) = h^{-1}(V) = h^{-1}(Y \cap U) = h^{-1}(Y) \cap h^{-1}(U) = h^{-1}(U)$ . So  $f^{-1}(V) \neq \phi$ . Since  $f$  is somewhat continuous, there exists  $\phi \neq W \subseteq X : W \subseteq f^{-1}(V)$  and  $\tau(W) \geq \delta(V) = \mu_Y(V) \geq \mu(V)$ . Thus  $W \subseteq h^{-1}(U)$  and  $\tau(W) \geq \mu(V)$ .

**Result 3.11** Let  $f : (X, \tau) \rightarrow (Y, \delta)$  be somewhat continuous and  $f(X) \subseteq Z \subseteq Y$ .

Then  $g : (X, \tau) \rightarrow (Z, \delta_Z)$  is somewhat continuous.

**Proof.** Let  $U \subseteq Z$  with  $g^{-1}(U) \neq \phi$ .

As  $f(X) \subseteq Z$ , then for any  $V \subseteq Y : U = Z \cap V, f^{-1}(V) = g^{-1}(U)$ .

As  $f$  is somewhat continuous, there exists  $\phi \neq W \subseteq X : W \subseteq f^{-1}(V)$  and  $\tau(W) \geq \delta(V)$ .

But this is true for any  $V \subseteq Y$  satisfying  $U = Z \cap V$ .

So  $\tau(W) \geq \vee \{\delta(V) : U = Z \cap V\} = \delta_Z(U)$ .

Hence there exists  $\phi \neq W \subseteq X : W \subseteq g^{-1}(U)$  and  $\tau(W) \geq \delta_Z(U)$ .

**Result 3.12** Let  $f : (X, \tau) \rightarrow (Y, \delta)$  and suppose  $A$  and  $B$  be subsets of  $X$  :

$X = A \cup B$  and  $\tau(A) = 1 = \tau(B)$ . If  $f_A : (A, \tau_A) \rightarrow (Y, \delta)$  and  $f_B : (B, \tau_B) \rightarrow (Y, \delta)$  are somewhat continuous then  $f$  is somewhat continuous.

**Proof.** Let  $U \subseteq Y$  and  $f^{-1}(U) \neq \emptyset$ . As  $f^{-1}(U) = f_A^{-1}(U) \cup f_B^{-1}(U)$ , either  $f_A^{-1}(U) \neq \emptyset$  or  $f_B^{-1}(U) \neq \emptyset$ . Suppose  $f_A^{-1}(U) \neq \emptyset$ . As  $f_A$  is somewhat continuous, there exists  $\emptyset \neq W \subseteq A : W \subseteq f_A^{-1}(U)$  and  $\tau_A(W) \geq \delta(U)$ . But  $\tau(A) = 1$  implies  $\tau_A(W) = \tau(W)$ .

Thus we have  $W \subseteq f_A^{-1}(U) \subseteq f^{-1}(U)$  and  $\tau(W) \geq \delta(U)$ .

**Definition 3.13** Let  $\tau$  and  $\delta$  be two fuzzified topologies on a set  $X$ .  $\tau$  is said to be weakly equivalent to  $\delta$  if for any  $\emptyset \neq W \subseteq X$ , there exists  $\emptyset \neq U, V \subseteq W$  such that  $\tau(U) \geq \delta(W)$  and  $\delta(V) \geq \tau(W)$ .

**Remark 3.14**  $\tau$  and  $\delta$  are weakly equivalent on  $X$  iff the identity function from  $(X, \tau)$  onto  $(X, \delta)$  is somewhat continuous in both directions

**Proof.** The proof is straightforward.

**Result 3.15** Let  $f : (X, \tau) \rightarrow (Y, \delta)$  be somewhat continuous. If  $\mu$  is weakly equivalent to  $\tau$ , then  $f : (X, \mu) \rightarrow (Y, \delta)$  is somewhat continuous.

**Proof.** Let  $U \subseteq Y$  and  $f^{-1}(U) \neq \emptyset$ . As  $f$  is somewhat continuous, there exists  $\emptyset \neq V \subseteq X : V \subseteq f^{-1}(U)$  and  $\tau(V) \geq \delta(U)$ .

Again  $\mu$  is weakly equivalent to  $\delta$ , therefore there exists  $\emptyset \neq W \subseteq V : \mu(W) \geq \tau(V)$ .

**Result 3.16** Let  $f : (X, \tau) \rightarrow (Y, \delta)$  be somewhat continuous and onto. If  $\delta$  is weakly equivalent to  $\nu$ , then  $f : (X, \tau) \rightarrow (Y, \nu)$  is somewhat continuous.

**Proof.** Let  $U \subseteq Y$  such that  $f^{-1}(U) \neq \emptyset$ . Then  $U \neq \emptyset$ . Again  $\delta$  is weakly equivalent to  $\nu$  implies there exists  $\emptyset \neq W \subseteq U : \delta(W) \geq \nu(U)$ . As  $f$  is onto  $W \neq \emptyset$  implies  $f^{-1}(W) \neq \emptyset$ .

Now  $f : (X, \tau) \rightarrow (Y, \delta)$  is somewhat continuous, therefore, there exists  $\emptyset \neq V \subseteq X$  such that  $V \subseteq f^{-1}(W)$  and  $\tau(V) \geq \delta(W)$ .

Thus there exists  $\emptyset \neq V \subseteq X$  such that  $V \subseteq f^{-1}(W)$  and  $\tau(V) \geq \nu(U)$ .

Combining Results 3.15 and 3.16 we have

**Result 3.17** Let  $f : (X, \tau) \rightarrow (Y, \delta)$  be somewhat continuous and onto. If  $\mu$  is weakly equivalent to  $\tau$  and  $\nu$  is weakly equivalent to  $\delta$ , then  $f : (X, \mu) \rightarrow (Y, \nu)$  is somewhat continuous.

**Definition 3.18** A fuzzified topological space is said to be separable if there exist a countable dense subset.

**Result 3.19** Let  $f : (X, \tau) \rightarrow (Y, \delta)$  be closed and somewhat continuous. If  $X$  is separable then so is  $Y$ .

**Proof.** Since image of a countable set is countable, the result follows from Result 3.7.

#### 4. SOMEWHAT OPEN FUNCTIONS IN FUZZIFIED TOPOLOGICAL SPACE

In this section we introduce and discuss the properties of somewhat open function in fuzzified topological spaces.

**Definition 4.1** Let  $(X, \tau)$  and  $(Y, \delta)$  be two FTSs. A function  $f : (X, \tau) \rightarrow (Y, \delta)$  is said to be somewhat open with respect to  $\tau$  and  $\delta$  if for each  $\phi \neq U \subseteq X$ , there exist  $\phi \neq V \subseteq X : V \subseteq f(U)$  and  $\delta(V) \geq \tau(U)$ .

It is clear from the definition that an open function is somewhat open. That the converse is not always true is evident from the following example.

**Example 4.2** Let  $X = \{a, b, c\}$ ,  $\tau$  and  $\mu$  are fuzzified topologies on  $X$  defined as :

	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	$\phi$	$X$
$\tau$	.1	.1	.1	.5	.1	.1	1	1
$\mu$	.6	.2	.2	.2	.2	.2	1	1

Let  $f : (X, \tau) \rightarrow (Y, \mu)$  be the identity function.

Then  $f$  is not open, since  $f(\{a, b\}) = \{a, b\}$  and  $\mu[f(\{a, b\})] = .2 < .5 = \tau\{a, b\}$ .

However  $f$  is somewhat open as there exists  $\{a\} \subseteq f(\{a, b\})$  such that

$$\mu(\{a\}) = .6 > .5 = \tau\{a, b\}$$

**Definition 4.3** Let  $(X, \tau)$  and  $(Y, \delta)$  be two FTSs. A function  $f : (X, \tau) \rightarrow (Y, \delta)$  is said to be somewhat closed with respect to  $\tau$  and  $\delta$  if for each  $U \subseteq X$ , there exist  $V \subseteq Y$  such that  $f(U) \subseteq V$  and  $\omega_\delta(V) \geq \omega_\tau(U)$ .

**Result 4.4** Let  $f : (X, \tau) \rightarrow (Y, \mu)$  is somewhat continuous, and  $g : (Y, \mu) \rightarrow (Z, \nu)$  is somewhat continuous, the  $(g \circ f) : (X, \tau) \rightarrow (Z, \nu)$ .

**Proof.** Let  $\phi \neq U \subseteq X$ . Consider  $(gof)(U)$ . Since  $f : (X, \tau) \rightarrow (Y, \mu)$  is somewhat continuous, there exist  $\phi \neq V \subseteq Y : V \subseteq f(U)$  and  $\mu(V) \geq \tau(U)$ . Again  $g : (Y, \mu) \rightarrow (Z, \nu)$  is somewhat continuous, so there exists  $\phi \neq W \subseteq Z : W \subseteq g(V)$  and  $\nu(W) \geq \mu(V)$ .

Thus we have  $\phi \neq W \subseteq g(f(U)) = (gof)(U)$  such that  $\nu(W) \geq \tau(U)$ .

**Result 4.5** Inverse image of a dense subset under a somewhat closed and continuous function is dense.

**Proof.** Let  $f : (X, \tau) \rightarrow (Y, \delta)$  be somewhat closed and continuous function. Let  $D$  be dense in  $Y$ . Consider  $f^{-1}(D)$ . Suppose  $f^{-1}(D)$  is not dense in  $X$ . Then there exists  $F \subsetneq X : f^{-1}(D) \subseteq F$  and  $\omega_\tau(F) > \omega_\tau(f^{-1}(D))$ . Now  $F \subsetneq X \Rightarrow f(F) \subsetneq Y$ . As  $f$  is somewhat closed, there exists  $B \subsetneq Y : f(F) \subseteq B$  and  $\omega_\delta(B) \geq \omega_\tau(F)$ .

Thus  $D \subseteq f(F) \subseteq B$  and  $\omega_\delta(B) \geq \omega_\tau(F) > \omega_\tau(f^{-1}(D)) \geq \omega_\delta(D)$ , since  $f$  is continuous. But this is contradiction to that fact that  $D$  is dense in  $Y$ . Hence  $f^{-1}(D)$  is dense in  $X$ .

**Result 4.6** Let  $f : (X, \tau) \rightarrow (Y, \delta)$  be one-one and onto. Then  $f$  is somewhat open iff  $f$  is somewhat closed.

**Proof.** Suppose  $f$  is open. Let  $U \subseteq X : f(U) \neq Y$ . To find  $V \subsetneq Y$  such that  $f(U) \subseteq V$  and  $\omega_\delta(V) \geq \omega_\tau(U)$ . If  $U = \phi$ , then  $f(U) = \phi$ . We choose  $V = \phi$ .

So let  $\phi \neq U \subsetneq X$ . Then  $\phi \neq f(U) \subsetneq Y$ . Let  $U^c = W$ , then  $\phi \neq W \subsetneq X$ . Since  $f$  is somewhat open, there exists  $\phi \neq V \subseteq Y :$

$$V \subseteq f(W) \text{ and } \tau(W) \geq \delta(V) \Rightarrow f(W)^c \subseteq V^c \text{ and } \omega_\tau(W^c) \geq \omega_\delta(V^c)$$

$\Rightarrow f(W^c) \subseteq V^c$  and  $\omega_\tau(U) \geq \omega_\delta(V^c)$ . Let  $V^c = F$ . Since  $\phi \neq V$ ,  $Y \neq V^c = F$ . Thus for  $U \subseteq X : f(U) \subsetneq Y$ , there exists  $F \subsetneq Y : f(U) \subseteq F$  and  $\omega_\tau(U) \geq \omega_\delta(F)$ . Hence  $f$  is somewhat closed.

Converse. Let  $\phi \neq U \subseteq X$ . If  $U = X$ ,  $f(U) = Y$ . Choosing  $V = Y$ , we are done.

So let  $\phi \neq U \subsetneq X$ . Let  $G = U^c$ , then  $\phi \neq G \subsetneq X$  and so  $f(G) \subsetneq Y$ , as  $f$  is onto.

Hence by hypothesis there exists  $W \subsetneq Y :$

$$f(G) \subseteq W \text{ and } \omega_\delta(W) \geq \omega_\tau(G) \Rightarrow W^c \subseteq f(G)^c \text{ and } \delta(W^c) \geq \tau(G^c)$$

$$\Rightarrow W^c \subseteq f(G^c) \text{ and } \delta(W^c) \geq \tau(G^c) \Rightarrow W^c \subseteq f(U) \text{ and } \delta(W^c) \geq \tau(U)$$

Let  $W^c = V$ . Then  $\phi \neq V \subseteq Y : V \subseteq f(U)$  and  $\delta(V) \geq \tau(U)$ . Hence  $f$  is somewhat open.

**Result 4.7** Let  $f: (X, \tau) \rightarrow (Y, \delta)$  and suppose  $A$  and  $B$  be subsets of  $X$ :

$X = A \cup B$ . If  $f: (A, \tau_A) \rightarrow (Y, \delta)$  and  $f: (B, \tau_B) \rightarrow (Y, \delta)$  are somewhat open then  $f$  is somewhat open.

**Proof.** Let  $\phi \neq U \subseteq X$ . If  $U \cap B = \phi$ , then  $f(U) = f_A(U)$ .

Since  $f_A$  is somewhat open, there exists  $\phi \neq V \subseteq Y: V \subseteq f_A(U)$  and  $\delta(V) \geq \tau_A(U)$ . But  $\tau_A(U) \geq \tau(U)$ . Therefore  $\phi \neq V \subseteq Y: V \subseteq f(U)$  and  $\delta(V) \geq \tau(U)$ .

Similar is the case if  $U \cap A = \phi$ . So let  $U \cap A \neq \phi \neq U \cap B$ . Then  $f(U) = f_A(U) \cup f_B(U)$ .

As  $f_A$  is somewhat open, there exists  $\phi \neq V \subseteq Y: V \subseteq f_A(U)$  and  $\delta(V) \geq \tau_A(U) \geq \tau(U)$ .

As  $f_B$  is somewhat open, there exists  $\phi \neq W \subseteq Y: W \subseteq f_B(U)$  and  $\delta(W) \geq \tau_B(U) \geq \tau(U)$ .

Then there exists  $\phi \neq F = V \cap W \subseteq Y$ :

$F \subseteq f_A(U) \cup f_B(U) = f(U)$  and  $\delta(F) = \delta(V \cap W) \geq \delta(V) \wedge \delta(W) \geq \tau(U)$ .

**Remark 4.8**  $\tau$  and  $\delta$  are weakly equivalent on  $X$  iff the identity function from  $(X, \tau)$  onto  $(X, \delta)$  is somewhat open in both directions.

**Result 4.9** Let  $f: (X, \tau) \rightarrow (Y, \delta)$  be somewhat open function. If  $\mu$  is weakly equivalent to  $\tau$  and  $\nu$  is weakly equivalent to  $\delta$ , then  $f: (X, \mu) \rightarrow (Y, \nu)$  is somewhat open function.

**Proof.** The proof is straightforward.

**Definition 4.10** A  $f: (X, \tau) \rightarrow (Y, \delta)$  is said to be somewhat homeomorphism if  $f$  is one-one, onto, somewhat open and somewhat continuous.

**Remark 4.11** If  $f$  is a somewhat homeomorphism then  $f^{-1}$  is also a somewhat homeomorphism.

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# ON A QUARTER-SYMMETRIC NON-METRIC CONNECTION IN A LORENTZIAN PARA-COSYMPLECTIC MANIFOLD

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**ABSTRACT :** In 1975, Golab introduced the notion of quarter symmetric connection in a Riemannian manifold with affine connection. This was further developed by Yano and Imai (1982), Rastogi (1978, 1987), Mishra and Pandey (1980), Mukhopadhyay, Ray and Barua (1980), Biwas and De (1997), Sengupta and Biswas (2003), Singh and Pandey (2007) and many others.

In this paper we define and study a quarter-symmetric non-metric connection on an LP-cosymplectic manifold. The curvature tensor and the Ricci tensor of the quarter-symmetric non-metric connection is found. A necessary and sufficient condition has been deduced for the Ricci tensor  $\bar{\nabla}$  to be symmetric and skew – symmetric under certain conditions. First and Second Bianchi identities associated with quarter-symmetric non-metric connection  $\bar{\nabla}$ . Einstein manifold, Weyl-conformal curvature tensor, special curvature tensor of a quarter-symmetric non-metric connection  $\bar{\nabla}$  is found.

## 1. INTRODUCTION

Let  $(M^n, g)$  be an  $n$ -dimensional differentiable manifold on which there are defined a tensor field  $\phi$  of type  $(1, 1)$ , a contravariant vector field  $\xi$ , a covariant vector field  $\eta$  and a Lorentzian metric  $g$ , which satisfy

$$\phi^2 X = X + \eta(X)\xi, \quad (1.1)$$

$$\eta(\xi) = -1, \quad (1.2)$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad (1.3)$$

$$\text{and} \quad g(X, \xi) = \eta(X), \quad (1.4)$$

then  $M^n$  is called a Lorentzian para-contact manifold (an LP-Contact manifold) and the structure  $(\phi, \xi, \eta, g)$  is called an LP-Contact structure (Matsumoto 1989).

In an LP-Contact Manifold, we have

$$(a) \phi\xi = 0 \quad (b) \eta(\phi X) = 0 \quad (c) \text{rank } \phi = (n - 1). \quad (1.5)$$

Let us put

$$F(X, Y) = g(\phi X, Y).$$

Then the tensor field  $F$  is symmetric (0, 2) tensor field

$$F(X, Y) = F(Y, X).$$

An LP-Contact manifold is said to be an LP-cosymplectic manifold (Prasad and Ojha, 1994) if

$$\nabla_X \phi = 0 \text{ implies } (\nabla_X F)(Y, Z) = 0. \quad (1.8)$$

On this manifold, we have

$$(\nabla_X \eta)(Y) = 0, \quad (1.9)$$

$$\text{and} \quad \nabla_X \xi = 0, \quad (1.10)$$

for vector fields  $X, Y$  and  $Z$ , where  $\nabla$  denotes covariant differentiation with respect to  $g$ .

Let  $(M^n, g)$  be an LP-cosymplectic manifold with Levi-Civita connection  $\nabla$ , we define a linear connection  $\bar{\nabla}$  on  $M^n$  by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X + \alpha(X)\phi Y. \quad (1.11)$$

where  $\eta$  and  $\alpha$  are 1-forms associated with vector field  $\xi$  and  $A$  on  $M^n$  given by

$$g(X, \xi) = \eta(X), \quad (1.12)$$

$$\text{and} \quad g(X, A) = \alpha(X), \quad (1.13)$$

for all vector fields  $X \in \chi(M^n)$ , where  $\chi(M^n)$  is the set of all differentiable vector fields on  $M^n$ .

Using (1.11), the torsion tensor  $\bar{T}$  of  $M^n$  with respect to the connection  $\bar{\nabla}$  is given by

$$\bar{T}(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y + \alpha(X)\phi Y - \alpha(Y)\phi X. \quad (1.14)$$

A linear connection satisfying (1.14) is called a quarter-symmetric connection.

Further using (1.11), we have

$$(\bar{\nabla}_X g)(Y, Z) = -\eta(Y)g(\phi X, Z) - \eta(Z)g(\phi X, Y) - 2\alpha(X)g(\phi Y, Z). \quad (1.15)$$

A linear connection  $\bar{\nabla}$  defined by (1.11) satisfies (1.14) and (1.15) is called a quarter-symmetric non-metric connection.

Let  $\bar{\nabla}$  be a linear connection in  $M^n$  given by

$$\bar{\nabla}_X Y = \nabla_X Y + H(X, Y). \quad (1.16)$$

Now, we shall determine the tensor field  $H$  such that  $\bar{\nabla}$  satisfies (1.14) and (1.15).

From (1.16), we have

$$\bar{T}(X, Y) = H(X, Y) - H(Y, X). \quad (1.17)$$

$$\text{Denote } G(X, Y, Z) \equiv (\bar{\nabla}_X g)(Y, Z) \quad (1.18)$$

From (1.16) and (1.18), we have

$$g(H(X, Y), Z) + g(H(X, Z), Y) = -G(X, Y, Z). \quad (1.19)$$

From (1.16), (1.18), (1.19) and (1.15), we have

$$\begin{aligned} & g(\bar{T}(X, Y), Z) + g(\bar{T}(Z, X), Y) + g(\bar{T}(Z, Y), X) \\ &= g(H(X, Y), Z) - g(H(Y, X), Z) + g(H(Z, X), Y) - g(H(X, Z), Y) \\ & \quad + g(H(Z, Y), X) - g(H(Y, Z), X) \\ &= 2g(H(X, Y), Z) + G(X, Y, Z) + G(Y, X, Z) - G(Z, X, Y). \\ &= 2g(H(X, Y), Z) - 2\eta(Z)g(\phi X, Y) - 2\alpha(X)g(\phi Y, Z) - 2\alpha(Y)g(\phi X, Z) + \\ & \quad 2\alpha(Z)g(\phi X, Y). \end{aligned}$$

$$\text{or, } H(X, Y) = \frac{1}{2} \{ \bar{T}(X, Y) + \bar{T}(Y, X) + \bar{T}(Y, X) \} + \alpha(X)\phi Y + \alpha(Y)\phi X + g(\phi X, Y)\xi - g(\phi X, Y)A.$$

Where  $T$  be a tensor field of type (1, 2) defined by .

$$g(\bar{T}(X, Y), Z) = g(\bar{T}(Z, X), Y)$$

$$\text{or, } H(X, Y) = \eta(Y)\phi X + \alpha(X)\phi Y.$$

This implies

$$\nabla_X Y = \bar{\nabla}_X Y + \eta(Y)\phi X + \alpha(X)\phi Y.$$

## 2. CURVATURE TENSOR OF AN LP-COSYMPLECTIC MANIFOLD WITH RESPECT TO THE QUARTER -SYMMETRIC NON-METRIC CONNECTION $\bar{\nabla}$

Let  $\bar{R}$  and  $R$  be the curvature tensor of the connection  $\bar{\nabla}$  and  $\nabla$  respectively then

$$\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X, Y]}Z. \quad (2.1)$$

From (1.11) and (2.1), we get

$$\begin{aligned}\bar{R}(X, Y)Z &= \bar{\nabla}_X(\nabla_Y Z) + \eta(Z)\phi Y + a(Y)\phi Z - \bar{\nabla}_Y(\nabla_X Z + \eta(Z)\phi X + a(X)\phi Z) \\ &\quad - \nabla_{[X, Y]}Z - \eta(Z)\phi([X, Y]) - a([X, Y])\phi Z.\end{aligned}\quad (2.2)$$

Using (1.8), (1.9) in (2.2), we get

$$\begin{aligned}R(X, Y)Z &= R(X, Y)Z + a(X)\eta(Z)Y - a(Y)\eta(Z)X + a(X)\eta(Y)\eta(Z)\xi \\ &\quad - a(Y)\eta(X)\eta(Z)\xi + da(X, Y)\phi Z.\end{aligned}\quad (2.3)$$

where  $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z$ ,

is the curvature tensor of  $\nabla$  with respect to the Riemannian connection.

Contracting (2.3), we find

$$\bar{S}(Y, Z) = S(Y, Z) - (n - 2)a(Y)\eta(Z) + a(\xi)\eta(Y)\eta(Z) + da(\phi Z, Y) \quad (2.4)$$

$$\text{and } \bar{r} = r - (n - 1)a(\xi) + \lambda \quad (2.5)$$

where  $\bar{S}$  and  $\bar{r}$  are the Ricci tensor and scalar curvature with respect to  $\bar{\nabla}$ ,

$\lambda = \text{trace } da(\phi Z, Y)$ .

Hence, we can state the following theorem.

**Theorem (2.1).** The curvature tensor  $\bar{R}(X, Y)Z$  the Ricci tensor  $\bar{S}(Y, Z)$  and the scalar curvature  $\bar{r}$  of an LP-cosymplectic manifold with respect to quarter-symmetric non-metric connection is given by (2.3), (2.4) and (2.5) respectively.

Let us assume that  $\bar{R}(X, Y)Z = 0$  in (2.3) and contracting, we get

$$S(Y, Z) = (n - 2)a(Y)\eta(Z) - a(\xi)\eta(Y)\eta(Z) - da(\phi Z, Y).$$

Which gives  $r = (n - 1)a(\xi) - \lambda$ .

Hence, we can state the following theorem:

**Theorem (2.2).** If an LP-cosymplectic manifold  $M^n$  admits a quarter-symmetric non-metric connection whose curvature tensor vanishes, then the scalar curvature  $r$  is given by

$$r = (n - 1)a(\xi) - \lambda.$$

From (2.3) it follows that

$$\bar{R}(X, Y, Z, W) + \bar{R}(Y, X, Z, W) = 0. \quad (2.6)$$

$$\begin{aligned} \text{and } \bar{R}(X, Y, Z, W) + \bar{R}(Y, Z, X, W) + \bar{R}(Z, X, Y, W) = & da(X, Y)g(\phi Z, W) + da(Y, Z) \\ & g(\phi X, W) + da(Z, X)g(\phi Y, W) + [a(X)\eta(Z) - a(Z)\eta(X)]g(Y, W) \\ & + [a(Z)\eta(Y) - a(Y)\eta(Z)]g(X, W) + [a(Y)\eta(X) - a(X)\eta(Y)]g(Z, W). \end{aligned} \quad (2.7)$$

Where  $\bar{R}(X, Y, Z, W) = g(\bar{R}(X, Y)Z, W)$

and  $\bar{R}(X, Y, Z, W) = g(R(X, Y)Z, W)$

Hence, we can state the following theorem:

**Theorem (2.3).** The curvature tensor of an LP-cosymplectic manifold with respect to the quarter-symmetric non-metric connection  $\bar{\nabla}$ , satisfies the relation (2.6) and (2.7)

### 3. SYMMETRIC AND SKEW-SYMMETRIC CONDITION OF RICCI TENSOR OF $\bar{\nabla}$ IN AN LP-COSYMPLECTIC MANIFOLD

From (2.4), we have

$$\bar{S}(Z, Y) = S(Z, Y) - (n-2)a(Z)\eta(Y) + a(\xi)\eta(Y)\eta(Z) + da(\phi Y, Z). \quad (3.1)$$

From (2.4) and (3.1), we have

$$\bar{S}(Y, Z) - \bar{S}(Z, Y) = (n-2)a(Z)\eta(Y) - (n-2)a(Y)\eta(Z) + da(\phi Z, Y) - da(\phi Y, Z). \quad (3.2)$$

If  $\bar{S}(Y, Z)$  is symmetric, then the L.H.S. of (3.2) vanishes, hence we get

$$(n-2)[a(Z)\eta(Y) - a(Y)\eta(Z)] = da(\phi Y, Z) - da(\phi Z, Y). \quad (3.3)$$

Moreover if relation (3.3) holds, then from (3.2),  $\bar{S}(Y, Z)$  is symmetric.

Hence, we can state the following theorem:

**Theorem (3.1).** The Ricci tensor  $\bar{S}(Y, Z)$  of the manifold with respect to the quarter-symmetric non-metric connection in an LP-cosymplectic manifold is symmetric if and only if the relation (3.3) holds.

Again from (3.4) and (3.1), we find

$$\begin{aligned} \bar{S}(Y, Z) + \bar{S}(Z, Y) = 2S(Y, Z) - (n-2)[a(Y)\eta(Z) + a(Z)\eta(Y) + \\ 2a(\xi)\eta(Y)\eta(Z) + da(\phi Z, Y) + da(\phi Y, Z)]. \end{aligned} \quad (3.4)$$

If  $\bar{S}(Y, Z)$  is skew-symmetric then the L.H.S. of (3.4) vanishes, and we get

$$\begin{aligned} S(Y, Z) = \frac{1}{2}(n-2)[a(Y)\eta(Z) + a(Z)\eta(Y)] - a(\xi)\eta(Y)\eta(Z) - \\ \frac{1}{2}[da(\phi Y, Z) + da(\phi Z, Y)]. \end{aligned} \quad (3.5)$$

Moreover if  $S(Y, Z)$  is given by (3.5), then from (3.4), we get

$$\bar{S}(Y, Z) + \bar{S}(Z, Y) = 0.$$

i.e. Ricci tensor of  $\bar{\nabla}$  is skew-symmetric.

Hence, we can state the following theorem:

**Theorem (3.2).** If an LP-cosymplectic manifold admits a quarter-symmetric non-metric connection  $\bar{\nabla}$  then a necessary and sufficient condition for the Ricci tensor of  $\bar{\nabla}$  to be skew-symmetric is that the Ricci tensor of the Levi-Civita connection  $\nabla$  is given by (3.5)

#### 4. BIANCHI FIRST AND SECOND IDENTITIES ASSOCIATED WITH QUARTER-SYMMETRIC NON-METRIC CONNECTION $\bar{\nabla}$ IN AN LP-COSYMPLECTIC MANIFOLD.

From (1.14), we have

$$\bar{T}(X, Y, Z) + \bar{T}(Y, Z, X) + \bar{T}(Z, X, Y) = 0, \quad (4.1)$$

where  $\bar{T}(X, Y, Z) = g(\bar{T}(X, Y), Z)$ .

Again from (1.14), we have

$$\begin{aligned} & \bar{T}(\bar{T}(X, Y), Z) + \bar{T}(\bar{T}(Y, Z), X) + \bar{T}(\bar{T}(Z, X), Y) \\ &= \eta(Y)\alpha(\phi X)\phi Z - \eta(X)\alpha(\phi Y)\phi Z + \alpha(X)\alpha(\phi Y)\phi Z - \alpha(Y)\alpha(\phi X)\phi Z + \\ & \quad \eta(Z)\alpha(\phi Y)\phi X - \eta(Y)\alpha(\phi Z)\phi X + \alpha(Y)\alpha(\phi Z)\phi X - \alpha(Z)\alpha(\phi Y)\phi X + \\ & \quad \eta(X)\alpha(\phi Z)\phi Y - \eta(Z)\alpha(\phi X)\phi Y + \alpha(Z)\alpha(\phi X)\phi Y - \alpha(X)\alpha(\phi Z)\phi Y. \end{aligned} \quad (4.2)$$

and

$$\begin{aligned} & (\bar{\nabla}_X \bar{T})(Y, Z) + (\bar{\nabla}_Y \bar{T})(Z, X) + (\bar{\nabla}_Z \bar{T})(X, Y) \\ &= da(X, Y)\phi Z + da(Y, Z)\phi X + da(Z, X)\phi Y \\ & \quad + \alpha(Z)\eta(Y)X - \alpha(Y)\eta(Z)X - \alpha(\phi X)\eta(Y)\phi Z - \alpha(X)\alpha(\phi Y)\phi Z + \alpha(\phi X)\eta(Z)\phi Y \\ & \quad + \alpha(X)\alpha(\phi Z)\phi Y + \alpha(X)\eta(Z)Y - \alpha(Z)\eta(X)Y - \alpha(\phi Y)\eta(Z)\phi X - \alpha(Y)\alpha(\phi Z)\phi X \\ & \quad + \alpha(\phi Y)\eta(X)\phi Z + \alpha(Y)\alpha(\phi X)\phi Z + \alpha(Y)\eta(X)Z - \alpha(X)\eta(Y)Z - \alpha(\phi Z)\eta(X)\phi Y \\ & \quad - \alpha(Z)\alpha(\phi X)\phi Y + \alpha(\phi Z)\eta(Y)\phi X + \alpha(Z)\alpha(\phi Y)\phi X. \end{aligned} \quad (4.3)$$

Bianchi first identity for a linear connection on  $M^n$  is given by (Sinha, 1982)

$$\begin{aligned} \bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y &= \bar{T}(\bar{T}(X, Y), Z) + \bar{T}(\bar{T}(Y, Z), X) \\ & \quad + \bar{T}(\bar{T}(Z, X), Y) + (\bar{\nabla}_X \bar{T})(Y, Z) + (\bar{\nabla}_Y \bar{T})(Z, X) + (\bar{\nabla}_Z \bar{T})(X, Y). \end{aligned} \quad (4.4)$$

Using (4.2) and (4.3) in (4.4), we get

$$\begin{aligned} \bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y &= da(X, Y)\phi Z + da(Y, Z)\phi X + da(Z, X)\phi Y + \\ \alpha(X)\eta(Z)Y - \alpha(X)\eta(Y)Z + \alpha(Y)\eta(X)Z &- \alpha(Y)\eta(Z)X + \alpha(Z)\eta(Y)X - \alpha(Z)\eta(X)Y. \end{aligned} \quad (4.5)$$

We call (4.5) as the first Bianchi's identity with respect to quarter-symmetric non-metric connection  $\bar{\nabla}$  in an LP-cosymplectic manifold.

This identity is also obtained by another way given by (2.7).

Bianchi's second identity for a linear connection on  $M^n$  is given by (Sinha, 1982)

$$\begin{aligned} (\bar{\nabla}_X \bar{Y})(Y, Z) + (\bar{\nabla}_Y \bar{X})(Z, X) + (\bar{\nabla}_Z \bar{R})(X, Y) \\ - \bar{R}(\bar{T}(X, Y), Z) - \bar{R}(\bar{T}(Y, Z), X) - \bar{R}(\bar{T}(Z, X), Y). \end{aligned} \quad (4.6)$$

Using (1.14) in above expression, we get

$$\begin{aligned} (\bar{\nabla}_X \bar{R})(Y, Z) + (\bar{\nabla}_Y \bar{R})(Z, X) + (\bar{\nabla}_Z \bar{R})(X, Y) - \eta(X)[\bar{R}(\phi Y, Z) - \bar{R}(\phi Z, Y)] \\ + \eta(Y)[\bar{R}(\phi Z, X) - \bar{R}(\phi X, Z)] + \eta(Z)[\bar{R}(\phi X, Y) - \bar{R}(\phi Y, X)] + \alpha(X)[\bar{R}(\phi Z, Y) \\ - \bar{R}(\phi Y, Z)] + \alpha(Y)[\bar{R}(\phi X, Z) - \bar{R}(\phi Z, X)] + \alpha(Z)[\bar{R}(\phi Y, X) - \bar{R}(\phi X, Y)]. \end{aligned} \quad (4.7)$$

we call (4.7) as second Bianchi's identity with respect to quarter-symmetric non-metric connection  $\bar{\nabla}$  in an LP-cosymplectic manifold.

Hence, we can state the following theorem :

**Theorem (4.1).** Bianchi's first and second identities associated with quarter-symmetric non-metric connection  $\bar{\nabla}$  in an LP-cosymplectic manifold is given by (4.5) and (4.7).

## 5. EINSTEIN MANIFOLD WITH RESPECT TO QUARTER - SYMMETRIC NON-METRIC CONNECTION $\bar{\nabla}$ IN AN LP-COSYMPLECTIC MANIFOLD

A Riemannian manifold  $M^n$  is called an Einstein manifold with respect to Riemannian connection if

$$S(X, Y) = \frac{r}{n} g(X, Y). \quad (5.1)$$

Analogous to this definition, we define Einstein manifold with respect to quarter-symmetric non-metric connection  $\bar{\nabla}$ .

$$\bar{S}(X, Y) = \frac{\bar{r}}{n} g(X, Y). \quad (5.2)$$

From (2.4), (2.5) and (5.2), we have

$$\begin{aligned} \bar{S}(X, Y) - \frac{\bar{r}}{n}g(X, Y) &= S(X, Y) - \frac{r}{n}g(X, Y) - (n-2)a(X)\eta(Y) \\ &+ a(\xi)\eta(X)\eta(Y) + da(\phi Y, X) - \left[ \frac{\lambda - (n-1)a(\xi)}{n} \right] g(X, Y). \end{aligned} \quad (5.3)$$

$$\text{If } da(\phi Y, X) - (n-2)a(X)\eta(Y) + a(\xi)\eta(X)\eta(Y) = \left[ \frac{\lambda - (n-1)a(\xi)}{n} \right] g(X, Y). \quad (5.4)$$

Then from (5.3), we get

$$\bar{S}(X, Y) - \frac{\bar{r}}{n}g(X, Y) = S(X, Y) - \frac{r}{n}g(X, Y). \quad (5.5)$$

Hence, we can state the following theorem.

**Theorem (5.1).** In an LP-cosymplectic manifold  $M^n$  with quarter-symmetric non-metric connection if the relation (5.4) holds, then the manifold is an Einstein manifold for the Riemannian connection if and only if it is an Einstein manifold for the connection  $\bar{\nabla}$ .

## 6. WEYL-CONFORMAL CURVATURE TENSOR OF A LP-COSYMPLECTIC MANIFOLD WITH RESPECT TO QUARTER - SYMMETRIC NON-METRIC CONNECTION

Weyl-conformal curvature tensor of a Riemannian manifold with respect to Riemannian connection is given by

$$\begin{aligned} {}^*C(X, Y, Z, W) &= {}^*R(X, Y, Z, W) - \frac{1}{(n-2)}[S(Y, Z)g(X, W) - S(X, Z)g(Y, W) \\ &+ S(X, W)g(Y, Z) - S(Y, W)g(X, Z)] + \frac{r}{(n-1)(n-2)}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \end{aligned} \quad (6.1)$$

Analogous to this definition, we define Weyl-conformal curvature tensor of  $M^n$  with respect to quarter-symmetric non-metric connection  $\bar{\nabla}$  given by

$$\begin{aligned} {}^*\bar{C}(X, Y, Z, W) &= {}^*\bar{R}(X, Y, Z, W) - \frac{1}{(n-2)}[\bar{S}(Y, Z)g(X, W) - \bar{S}(X, Z)g(Y, W) \\ &+ \bar{S}(X, W)g(Y, Z) - \bar{S}(Y, W)g(X, Z)] + \frac{\bar{r}}{(n-1)(n-2)}[g(Y, Z)g(X, W) \\ &- g(X, Z)g(Y, W)]. \end{aligned} \quad (6.2)$$



Where  $\text{'}C(X, Y, Z, W) = g(C(X, Y)Z, W)$ ,

and  $\text{'}\overline{C}(X, Y, Z, W) = g(\overline{C}(X, Y)Z, W)$ .

In consequence of (2.3), (2.4), (2.5), (6.1) and (6.2), we find

$$\begin{aligned} \text{'}\overline{C}(X, Y, Z, W) &= \text{'}C(X, Y, Z, W) + a(X)\eta(Y)\eta(Z)\eta(W) - a(Y)\eta(X)\eta(Z)\eta(W) + a(X)\eta(W) \\ &\quad g(Y, Z) - a(Y)\eta(W)g(X, Z) + da(X, Y)g(\eta Z, W) - \frac{1}{(n-2)} [a(\xi)\eta(Y)\eta(Z) \\ &\quad g(X, W) - a(\xi)\eta(Z)g(Y, W) + a(\xi)\eta(X)\eta(W)g(Y, Z) - a(\xi)\eta(Y)\eta(W)g(X, Z) \\ &\quad + da(\phi Z, Y)g(X, W) - da(\phi Z, X)g(Y, W) + da(\phi W, X)g(Y, Z) - da(\phi W, Y)g(X, Z)] \\ &\quad + \frac{[\lambda - (n-1)a(\xi)]}{(n-1)(n-2)} [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]. \end{aligned} \quad (6.3)$$

From (6.3), we have

$$\text{'}\overline{C}(X, Y, Z, W) + \text{'}\overline{C}(Y, X, Z, W) = 0 \quad (6.4)$$

and

$$\begin{aligned} \text{'}\overline{C}(X, Y, Z, W) + \text{'}\overline{C}(Y, Z, X, W) + \text{'}\overline{C}(Z, X, Y, W) \\ = da(X, Y)g(\phi Z, W) + da(Y, Z)g(\phi X, W) + da(Z, X)g(\phi Y, W) + \\ [da(\phi Z, Y) - da(\phi Y, Z)]g(X, W) + [da(\phi X, Z) - da(\phi Z, X)]g(Y, W) + \\ [da(\phi Y, X) - da(\phi X, Y)]g(Z, W). \end{aligned} \quad (6.5)$$

If the 1-form  $a$  is closed then from (6.5) it follows that

$$\text{'}\overline{C}(X, Y, Z, W) + \text{'}\overline{C}(Y, Z, X, W) + \text{'}\overline{C}(Z, X, Y, W) = 0 \quad (6.6)$$

Hence, we can state the following theorem:

**Theorem (6.1).** The Weyl-conformal curvature tensor of an LP-cosymplectic manifold with respect to quarter-symmetric non-metric connection is given by (6.3). It satisfies the relation (6.4) and (6.5). In particular, if the 1-form  $a$  is closed, then

$$\text{'}\overline{C}(X, Y, Z, W) + \text{'}\overline{C}(Y, Z, X, W) + \text{'}\overline{C}(Z, X, Y, W) = 0.$$

Let us assume that the curvature tensor of the quarter-symmetric non-metric connection  $\overline{\nabla}$  has the form

$$\begin{aligned}\bar{R}(X, Y)Z &= \alpha(X)\eta(Z)Y - \alpha(Y)\eta(Z)X + \alpha(X)\eta(Y)\eta(Z)\xi - \alpha(Y)\eta(X)\eta(Z)\xi + \\ &\quad \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + da(X, Y)\phi Z.\end{aligned}\quad (6.7)$$

Then (2.3) becomes

$$R(X, Y)Z = \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X. \quad (6.8)$$

Which gives

$$S(Y, Z) = g(Y, Z) - (n - 2)\eta(Y)\eta(Z). \quad (6.9)$$

$$\text{and} \quad r = 2(n - 1). \quad (6.10)$$

Using (6.8), (6.9) and (6.10) in (6.1), we get

$$C(X, Y)Z = 0.$$

Hence, we can state the following theorem:

**Theorem (6.2).** If an LP-cosymplectic manifold  $M^n$  admits a quarter-symmetric non-metric connection whose curvature tensor is of the form (6.7), then  $M^n$  is Weyl-conformally flat.

## 7. SPECIAL CURVATURE TENSOR OF AN LP-COSYMPLECTIC MANIFOLD WITH RESPECT TO QUARTER-SYMMETRIC NON-METRIC CONNECTION

Recently Singh and Khan in 1998 define a Special curvature tensor of the type (1.3) by the relation.

$$J(X, Y)Z = R(X, Y)Z + R(X, Z)Y. \quad (7.1)$$

It is obvious that

$$J(X, Y)Z = J(X, Z)Y. \quad (7.2)$$

$$\text{and} \quad J(X, Y)Z + J(Y, Z)X + J(Z, X)Y = 0 \quad (7.3)$$

Analogous to this definition, we define special curvature tensor of  $M^n$  with respect to quarter-symmetric non-metric connection  $\bar{\nabla}$  given by

$$\bar{J}(X, Y)Z = \bar{R}(X, Y)Z + \bar{R}(X, Z)Y. \quad (7.4)$$

Using (2.3), (7.1) and (7.2) in (7.4), we find

$$\begin{aligned}\bar{J}(X, Y)Z &= J(X, Y)Z + \alpha(X)\eta(Z)Y - \alpha(Y)\eta(Z)X + \alpha(X)\eta(Y)Z - \alpha(Z)\eta(Y) \\ &\quad X + 2\alpha(X)\eta(Y)\eta(Z)\xi - \alpha(Y)\eta(X)\eta(Z)\xi - \alpha(Z)\eta(X)\eta(Y)\xi + \\ &\quad da(X, Y)\phi Z + da(X, Z)\phi Y.\end{aligned}\quad (7.5)$$

From (7.5), we have

$$\bar{J}(X, Y)Z = \bar{J}(X, Z)Y. \quad (7.6)$$

and 
$$\bar{J}(X, Y)Z + \bar{J}(Y, Z)X + \bar{J}(Z, X)Y = 0. \quad (7.7)$$

Hence, we can state the following theorem:

**Theorem (7.1).** The special curvature tensor  $\bar{J}$  of an LP-cosymplectic manifold with respect to quarter-symmetric non-metric connection satisfies the following relation

$$\bar{J}(X, Y)Z = \bar{J}(X, Z)Y. \quad (7.8)$$

$$\bar{J}(X, Y)Z + \bar{J}(Y, Z)X + \bar{J}(Z, X)Y = 0. \quad (7.9)$$

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## ON FUZZY SOCLE OF MODULES

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**ABSTRACT :** Our attempt is to investigate the fuzzy aspects of socle of modules over rings. In this paper we define fuzzy simple submodules and fuzzy socle of modules. Using the notions of fuzzy essential submodules and fuzzy relative complement various results related to fuzzy socles of modules are established.

**Key words :** Fuzzy submodule, fuzzy socle, fuzzy essential submodule.

### INTRODUCTION

In 1965, Lotfi Zadeh introduced the concept of fuzzy set [12], and it was a new episode towards the development of science and engineering. In the trajectory of stupendous growth of fuzzy set theory, fuzzy algebra has become an important area of research. In 1971, A. Rosenfeld used the concept of Zadeh in abstract algebra [8] and opened up a new insight in the field of Mathematical science. Since then many researchers are working on the concepts like fuzzy semigroups, fuzzy groups, fuzzy rings, fuzzy semirings, fuzzy near-rings and so on. W. Liu [3] initiated the study of fuzzy subrings and fuzzy ideals of rings around 1982. Fuzzy submodules of a module  $M$  over a ring  $R$  were first introduced by Negoita and Relesco [6], Pan [7] studied fuzzy finitely generated modules and quotient modules. Mukherjee et al [5], kumar et al [1,2] studied various aspects of fuzzy submodules. In [11] Sidky introduced the notion of radical of a fuzzy submodule and also defined primary fuzzy submodules and obtained some important properties. Our attempt is to investigate the fuzzy aspects of socle of a module. Using the concepts of fuzzy essentiality and relative complement defined by Saikia et al in [10], it is proved that if  $\mu$  is a fuzzy submodule of  $M$  such that  $\mu = \text{Soc}(\mu)$  then  $\mu$  has no proper fuzzy essential submodules. If  $\mu$  has no proper fuzzy essential submodules then  $L(\mu)$  is complemented and  $\mu = \text{Soc}(\mu)$ . It is shown that if  $\xi$  is the intersection of all fuzzy essential submodules of  $\mu$ , where  $\mu$  is a fuzzy submodule of  $M$  then every non zero fuzzy submodule of  $\xi$  contains a simple fuzzy submodule of  $\xi$ . It leads us to the result that  $\text{soc}(\mu) = \xi$ .

### 1. DEFINITIONS AND NOTATIONS

Throughout this paper  $R$  denotes a commutative ring with unity and  $M$  denotes a module over  $R$ .

A fuzzy subset  $\mu$  of  $M$  is called a *fuzzy submodule* of  $M$  if the following conditions are satisfied:

- (i)  $\mu(x - y) \geq \mu(x) \wedge \mu(y)$ , for all  $x, y \in M$
- (ii)  $\mu(rx) \geq \mu(x)$ , for all  $r \in R, x \in M$ .
- (iii)  $\mu(0) = 1$

Let  $\mu, \sigma$  be two fuzzy submodules of  $M$ . If  $\sigma \subseteq \mu$  then  $\sigma$  is called a *fuzzy submodule* of  $\mu$ .

The collection of all fuzzy submodules of  $M$  is denoted by  $L(M)$  and if  $\mu \in L(M)$  then  $L(\mu)$  denotes the family of all submodules of  $\mu$ .

Let  $\mu$  be a fuzzy subset of a non empty set  $X$ . Then a *fuzzy point*  $x_t, x \in X, t \in (0, 1]$  is defined as the fuzzy subset  $x_t$  of  $X$  such that  $x_t(x) = t$ , and  $x_t(y) = 0$ , for all  $y \in X - \{x\}$ . We write  $x_t \in \mu$  if and only if  $x \in \mu_t$ .

Let  $\mu$  be a fuzzy subset of a non empty set  $X$ . Then support of  $\mu$ , denoted by  $\mu^*$ , is defined as  $\mu^* = \{x \in X : \mu(x) > 0\}$ . If  $\mu \in L(M)$  then  $\mu^*$  is a submodule of  $M$ .

A fuzzy submodule  $\mu$  of  $M$  is called a *fuzzy essential submodule* of  $M$ , denoted by  $\mu \subseteq_e M$  if for every nonzero fuzzy submodule  $\theta$  of  $M$ ,  $\mu \cap \theta \neq \chi_0$ .

Let  $\mu$  and  $\sigma$  be two non zero fuzzy submodules of  $M$  such that  $\mu \subseteq \sigma$ . Then  $\mu$  is called *fuzzy essential* in  $\sigma$ , denoted by  $\mu \subseteq_e \sigma$ , if for every non zero fuzzy submodule  $\nu$  of  $M$  satisfying  $\nu \subseteq \sigma$ ,  $\mu \cap \nu \neq \chi_0$ .

Let  $\mu$  be a fuzzy submodule of  $M$ . A *relative complement* for  $\mu$  in  $M$  is any fuzzy submodule  $\sigma$  of  $M$  which is maximal with respect to the property  $\mu \cap \sigma = \chi_0$ .

Let  $\mu$  be a fuzzy submodule of a fuzzy submodule  $\delta$ . A *relative complement* for  $\mu$  in  $\delta$  is which is maximal with respect to the property  $\mu \cap \sigma = \chi_0$ .

A fuzzy submodule  $\theta$  of  $M$  is said to be *fuzzy simple submodule* if  $\mu \subseteq \theta$  where  $\mu \in L(M)$  implies either  $\mu = \chi_0$  or  $\mu = \theta$ .

If  $\mu$  is a fuzzy submodule of  $M$  then *socle* of  $\mu$ , denoted by  $\text{Soc}(\mu)$ , is defined as the sum of all simple fuzzy submodules of  $\mu$ . Thus  $\text{Soc}(\mu) = \sum \theta_i$ , where  $\theta_i$  is a fuzzy simple submodule of  $\mu$ . If  $\mu$  has no fuzzy simple submodules then  $\text{Soc}(\mu) = \chi_0$ .

If  $\mu \in L(M)$  then  $L(\mu) = \{\theta \subseteq \mu : \theta \in L(M)\}$  is fuzzy complemented if for all  $\theta \subseteq \mu$ ,  $\theta \in L(M)$  there exists  $\theta' \in L(M)$  such that  $\theta \cap \theta' = \chi_0$  and  $\theta + \theta' = \mu$ .

If  $\mu, \theta \in L(M)$  then direct sum of  $\mu$  and  $\theta$ , denoted by  $\mu \oplus \theta$  is defined as  $\mu \oplus \theta = \mu + \theta$  provided  $\mu \cap \theta = \chi_0$ . If  $\sigma, \mu, \theta \in L(M)$  be such that  $\sigma = \mu \oplus \theta$  then  $\mu$  is called a *fuzzy direct summand* of  $\sigma$ .

## 2. PRELIMINARIES

Now we present the preliminary results that are needed in the sequel.

**Lemma 2.1[11]:** Let  $\mu$  be a fuzzy subset of  $M$ . Then the level subset  $\mu_t = \{x \in M : \mu(x) \geq t\}$ ,  $t \in \text{Im } \mu$  is a submodule of  $M$  if and only if  $\mu$  is a fuzzy submodule of  $M$ .

We state the following results which are proved in [10]

**Lemma 2.2:** Let  $\mu$  be an essential fuzzy submodule of  $M$ . Then  $\mu_t$  is an essential submodule of  $M$ , for some  $t \in \text{Im } \mu$ .

**Lemma 2.3 :** Every fuzzy submodule of  $M$  is an essential fuzzy submodule of itself.

**Lemma 2.4:** Let  $\mu_1, \mu_2, \sigma_1, \sigma_2 \in L(M)$  be such that  $\mu_1 \subseteq_e \sigma_1$  and  $\mu_2 \subseteq_e \sigma_2$ . Then  $\mu_1 \cap \mu_2 \subseteq_e \sigma_1 \cap \sigma_2$ .

**Lemma 2.5:** Let  $\mu$  and  $\sigma$  be two non zero fuzzy submodules of  $M$  such that  $\mu \subseteq_e \sigma$ . Then for any  $\theta \in L(M)$ ,  $\mu \cap \theta \subseteq_e \sigma \cap \theta$ .

**Lemma 2.6:** Let  $\mu, \nu$  and  $\sigma$  be non zero fuzzy submodules of  $M$  such that  $\mu \subseteq \nu \subseteq \sigma$ . then  $\mu \subseteq_e \sigma$  if and only if  $\mu \subseteq_e \nu \subseteq_e \sigma$ .

**Lemma 2.7:** Let  $\mu, \gamma \in L(M)$  such that  $\mu \cap \gamma = \chi_0$ . Then  $\gamma$  can be enlarged to a relative complement for  $\mu$ .

**Lemma 2.8:** Let  $\gamma, \sigma, \theta \in L(M)$  such that  $\sigma \subseteq \theta$ . Then  $\theta \cap (\gamma + \sigma) = (\theta \cap \gamma) + \sigma$

**Proof:** Let  $\mu = \theta \cap (\gamma + \sigma)$ . Then  $\mu^* = (\theta \cap (\gamma + \sigma))^* = \theta^* \cap (\gamma^* + \sigma^*)$ , by theorem 3.3.7 in [4] Since  $\sigma \subseteq \theta$ , so  $\sigma^* \subseteq \theta^*$ . Now by modular law  $\theta^* \cap (\gamma^* + \sigma^*) = (\theta^* \cap \gamma^*) + \sigma^*$ . Thus  $\mu^* = (\theta \cap \gamma)^* + \sigma^*$ . Also by theorem 3.3.7 in [4], we get  $\mu = (\theta \cap \gamma) + \sigma$ . Thus the result follows.

## 3. MAIN RESULTS

We now present our main results

**Theorem 3.1:** If  $\mu = \text{Soc}(\mu)$  then  $\mu$  has no proper fuzzy essential submodules.

**Proof:** Given  $\mu = \text{Soc}(\mu) = \sum \theta_i$ ,  $\theta_i$  is a fuzzy simple submodule of  $\mu$ . Let  $\alpha$  be any fuzzy

essential submodule of  $\mu$ . Then there exists a fuzzy submodule  $\theta'_i$  of  $\mu$  such that  $\alpha \cap \theta_i = \theta'_i$  and  $\theta'_i \neq \chi_0$ .

Since  $\theta'_i(x) = (\alpha \cap \theta_i)(x) = \alpha(x) \wedge \theta_i(x) \leq \alpha(x)$ , so  $\theta'_i \subseteq \alpha$ . Similarly  $\theta'_i \subseteq \theta_i$ .

Now  $\theta_i$  is a fuzzy simple submodule of  $\mu$ , so  $\theta'_i \subseteq \theta_i$  implies  $\theta'_i = \theta_i$ . Also  $\theta'_i \subseteq \alpha$  implies that  $\alpha$  contains all fuzzy simple submodule of  $\mu$ . Thus  $\text{Soc}(\mu) \subseteq \alpha$ . This gives  $\mu \subseteq \alpha$ . Thus  $\mu = \alpha$ . By lemma 2.3,  $\mu$  is an essential submodule of itself. Hence  $\mu$  has no proper fuzzy essential submodules.

**Theorem 3.2:** If  $\mu \in L(M)$  and  $\xi$  is the intersection of all fuzzy essential submodules of  $\mu$  then  $\text{soc}(\mu) \subseteq \xi$ .

**Proof:** Let  $\sigma, \theta \in L(M)$  be such that  $\theta$  is a fuzzy simple submodule of  $\mu$  and  $\sigma \subseteq_e \mu$ . Then  $\sigma \cap \theta \neq \chi_0$ . Also  $\sigma \cap \theta \subseteq \theta$  and  $\theta$  being a fuzzy simple submodule we have  $\sigma \cap \theta = \theta$ . Now  $\theta(x) = (\sigma \cap \theta)(x) = \sigma(x) \wedge \theta(x) \leq \sigma(x)$ . Thus  $\theta \subseteq \sigma$ . This implies that if  $\theta$  is a fuzzy simple submodule of  $\mu$  then  $\theta$  is contained in every fuzzy essential submodule of  $\mu$ . Hence  $\text{soc}(\mu) \subseteq \xi$ .

**Theorem 3.3:** Let  $\mu \in L(M)$  and  $\xi$  be the intersection of all fuzzy essential submodules of  $\mu$ . If every non-zero fuzzy submodule of  $\xi$  is a direct summand of  $\xi$ , then every nonzero fuzzy submodule of  $\xi$  contains a fuzzy simple submodule of  $\mu$ .

**Proof:** Let  $\theta (\neq \chi_0) \in L(M)$  be such that  $\theta \subseteq \xi$ . We consider  $\mathcal{F} = \{\mu : \mu \subseteq \theta, \mu \in L(M)\}$ . By Zorn's lemma there exists a maximal element  $\sigma$  in  $\mathcal{F}$  such that  $\sigma \subseteq \theta$  and  $\sigma \subseteq L(M)$ . By the given condition  $\xi = \sigma \oplus \sigma'$ , for some  $\sigma' \in L(M)$ .

Now  $\theta = \theta \cap \xi = \theta \cap (\sigma \oplus \sigma') = \sigma \oplus (\theta \cap \sigma')$ , by lemma 2.8. If  $\theta \cap \sigma'$  is not a fuzzy simple submodule then it contains a non zero fuzzy submodule  $v$  of  $M$ . So there exists  $v' \in L(M)$  such that  $\xi = v \oplus v'$ . Also  $\theta \cap \sigma' = (\theta \cap \sigma') \cap \xi = (\theta \cap \sigma') \cap (v \oplus v') = v' \oplus (\theta \cap \sigma' \cap v)$ . This implies  $\sigma \oplus (\theta \cap \sigma') = \sigma \oplus v' \oplus (\theta \cap \sigma' \cap v) = \sigma \oplus v$ . Thus  $\theta \cap \sigma' = v$ , which is a contradiction. Therefore  $\theta \cap \sigma'$  is a fuzzy simple submodule. Thus the result follows.

**Theorem 3.4:** If  $\mu \in L(M)$  and  $\xi$  is the intersection of all fuzzy essential submodules of  $\mu$  then  $\xi \subseteq \text{soc}(\mu)$ .

**Proof:** Let  $\theta$  be a fuzzy submodule of  $\xi$ . Then  $\theta$  is a fuzzy submodule of  $\mu$ . So there exists a fuzzy submodule  $\sigma$  such that  $\sigma$  is a relative complement of  $\theta$  in  $\mu$ . Let  $\theta' \in L(M)$  be such that  $\theta' \cap (\theta \oplus \sigma) = \chi_0$ . Now  $\theta \subseteq \theta \oplus \sigma$  implies  $\theta \cap \theta' = \chi_0$ . Similarly  $\sigma \cap \theta' = \chi_0$ . If  $\theta \cap (\sigma \oplus \theta') \neq \chi_0$ , then there exists a non zero element  $x$  in  $M$  such that  $(\theta \cap (\sigma \oplus \theta'))(x)$



$\geq t$ , where  $t \in (0, 1]$ . Thus  $\theta(x) \geq t$  and  $(\sigma \oplus \theta')(x) \geq t$ . This implies there exists some unique,  $y, z$  in  $M$  such that  $x = y + z$  and  $\sigma(y) \wedge \theta'(z) \geq t$  with  $\sigma(y) > 0, \theta'(z) > 0$ . Thus  $x = y + z$  with  $x \in \theta^*, y \in \sigma^*, z \in \theta'^*$ . Also  $z$  is a nonzero element of  $M$ , for otherwise it will imply that  $x$  is the zero element of  $M$ . Now,  $z = x - y \in \theta'^* \cap (\sigma^* \oplus \theta^*) = (\theta' \cap (\sigma \oplus \theta))^*$ . This shows  $\theta' \cap (\sigma \oplus \theta) \neq \chi_0$ , a contradiction. Thus  $\theta \cap (\sigma \oplus \theta') = \chi_0$ . By maximality of  $\sigma$ ,  $\sigma \oplus \theta' = \sigma$ . Now  $\sigma(x) = (\sigma \oplus \theta')(x) \geq \sigma(0) \wedge \theta'(x) = \theta'(x)$ . Thus  $\theta' \subseteq \sigma$  and hence  $\chi_0 = \theta' \cap \sigma = \theta'$ . This proves  $\theta \oplus \sigma \subseteq_e \mu$ . Thus  $\xi \subseteq \theta \oplus \sigma$ . This implies  $\xi = \xi \cap (\theta \oplus \sigma) = \xi \oplus (\theta \cap \sigma)$ , since  $\theta \subseteq \xi$  and  $\theta \subseteq \xi$  and  $\theta \cap (\xi \cap \sigma) = \chi_0$ . Thus every fuzzy submodule of  $\xi$  is a direct summand.

Let  $v$  be the sum of all fuzzy simple submodules of  $\xi$ . Then  $v$  is a direct summand of  $\xi$ . So there exists  $v' \subseteq L(M)$  such that  $\xi = v \oplus v'$ . If  $v' \neq \chi_0$  then there exists a fuzzy simple submodule  $\gamma$  of  $v'$ . This gives  $\gamma \subseteq v$ , a contradiction. Thus  $v' = \chi_0$ . This implies  $\xi = v$ . Hence  $\xi \subseteq \text{soc}(\mu)$ .

Using theorem 3.2, 3.3 and 3.4 we get the following

**Theorem 3.5:** If  $\mu \in L(M)$  and  $\xi$  is the intersection of all fuzzy essential submodules of  $\mu$  then  $\text{soc}(\mu) = \xi$ .

**Theorem 3.6:** If  $\mu = \text{soc}(\mu)$  then  $L(\mu)$  is fuzzy complemented and for any  $\theta \in L(\mu)$ ,  $L(\theta)$  is fuzzy complemented.

**Proof:** By theorem 3.1,  $\mu$  has no proper fuzzy essential submodules. Let  $\theta$  be any fuzzy submodule of  $\mu$ . If  $\sigma$  is a relative complement for  $\theta$  in  $\mu$  then as in the above theorem we get  $\sigma \oplus \theta \subseteq_e \mu$ . But given that  $\mu$  has no proper fuzzy essential submodules, So  $\sigma + \theta = \mu$  and  $\sigma$  being a relative complement for  $\theta$  we get  $\sigma \cap \theta = \chi_0$ . Hence  $L(\mu)$  is fuzzy complemented.

Let  $\sigma \in L(M)$  and  $\sigma \subseteq \theta$ . Then  $\sigma \subseteq \mu$ . As  $L(\mu)$  is fuzzy complemented, so there exists  $\gamma \in L(M)$  such that  $\sigma + \gamma = \mu$  and  $\sigma \cap \gamma = \chi_0$ . Now  $(\theta \cap \gamma) \cap \sigma = \theta \cap (\gamma \cap \sigma) = \theta \cap \chi_0 = \chi_0$ . Also by lemma 2.8,  $(\theta \cap \gamma) + \sigma = \theta \cap (\gamma + \sigma) = \theta \cap \mu = \theta$ . Hence there exists  $\theta \cap \gamma (\subseteq \theta) \in L(M)$  such that  $(\theta \cap \gamma) \cap \sigma = \chi_0$  and  $(\theta \cap \gamma) + \sigma = \theta$ . Thus  $L(\theta)$  is fuzzy complemented.

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## A GENERALIZATION OF EKELAND'S VARIATIONAL PRINCIPLE

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**ABSTRACT :** Ekeland's variational principle, and its equivalence theorem, Caristi's fixed point theorem are well known results in non linear analysis and have extensive application in various fields in mathematics, such as in control theory global analysis, and geometric theory of Banach space. In this paper, we prove few existence results for a non Hausdorff space with weak completeness notion. As an application, the Ekeland's Variational Principle is extended to a left (right) P-sequentially complete  $T_0$  quasi gauge space, which unifies many results. Illustrative and counter examples are also given.

**Key words :** W-distance; Quasi-order  $\prec$ ; Qusi-Gauge space; Lower semi continuous; Ekeland's variational principle; The Caristi-Kirk-Browder fixed point theorem; Takahashi's non-convex minimization theorem.

Ekeland's variational principle [4] and its equivalence theorem, Caristi's fixed point theorem [2] are well known results in non linear analysis and have extensive applications, in various fields in mathematics, such as in control theory global analysis, and geometric theory of Banach space. Many extensions or equivalent formulations of the principle appear as seen in the reference.

In this paper, the Ekeland's Variational Principle is extended to a left (right) P-sequentially complete  $T_0$  quasi gauge space. The extended form of the W-distance defined in [8] is used. Kada et al [8] improved Takahashi's non-convex minimization theorem [15] replacing the involved metric by the W-distance function. Recently S Park & Kang [14] unified extensions in a general form. Along these lines S Park [13] unified the result of Blum & Oettli [1], Kadatal [8], Oettli and Thera [12]. In fact they obtained yet another extended form of the principle and improved the equivalent formulations of Ekeland's variational principle with few applications. The aim of this paper is to further generalize this result to a  $T_0$  space with weak completeness notion.

We need following definitions.

### QUASI-GAUGE SPACE

A quasi-pseudo metric on a set  $X$  is a non-negative real valued function on  $X \times X$  such that for any  $x, y, z \in X$ .

$$p(x, x) = 0 \text{ for all } x \in X \text{ and } p(x, y) \leq p(x, z) + p(z, y)$$

A quasi-gauge structure for a topological space is a family  $P$  of quasi-pseudo metrics on  $X$  such that  $T$  has a sub-base the family  $\{B(x, p, \varepsilon): x \in X, p \in P, \varepsilon > 0\}$  where  $B(x, p, \varepsilon)$  is the set  $\{y \in X: p(x, y) < \varepsilon\}$ . If a topological space  $(X, T)$  has a quasi-gauge structure  $P$ , then it is called a quasi-gauge space. Every topological space is a quasi-gauge space. The sequence  $\{x_n\}$  in  $X$  is called left (right)  $P$ -Cauchy sequence if for each  $p$  in  $P$  and each  $\varepsilon > 0$  there is a point  $x$  in  $X$  and an integer  $k$  such that

$$p(x, x_m) < \varepsilon [p(x_m, x) < \varepsilon] \text{ for all } m \geq k. (x \text{ and } k \text{ may depend on } \varepsilon \text{ and } p).$$

A quasi-gauge space  $(X, P)$  is left (right)  $P$ -sequentially complete if every left (right)  $P$ -Cauchy sequence in  $X$  converges to some element of  $X$ .  $X$  is a  $T_0$  space iff

$$p(x, y) = p(y, x) = 0 \text{ for all } p \text{ in } P \text{ implies } x = y.$$

$$X \text{ is a } T_1 \text{ space iff } p(x, y) = 0 \text{ for all } p \text{ in } P \text{ implies } x = y.$$

We extend the definition of W-distance defined on a metric space  $(X, d)$  [8] to a quasi-gauge space  $(X, P)$  as follows.

**Definition.** A function  $\omega : X \times X \rightarrow [0, \infty)$  is called a W-distance on  $X$  if the following are satisfied;

1.  $\omega(x, z) \leq \omega(x, y) + \omega(y, z)$  for any  $x, y, z \in X$
2.  $\omega(x): X \rightarrow [0, \infty)$  is lower semi continuous for any  $x \in X$ ; and
3. for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $\omega(z, x) \leq \delta$  and  $\omega(z, y) \leq \delta$  imply  $p(x, y) \leq \varepsilon$  for each  $p \in P$

In [8], many examples and properties of W-distance were given

Let  $X$  be a non-empty set and ' $\preceq$ ' a quasi-order (pre order or pseudo-order, that is a reflexive and transitive relation) on  $X$

Let  $S(x) = \{y \in X: x \preceq y\}$  for  $x \in X$  and  $\leq$  be the usual order in the extended real number system  $[-\infty, \infty]$

In a quasi gauge space  $(X, P)$  with quasiorder ' $\preceq$ ' a set  $S(u)$  for some  $u \in X$  is said to be  $\preceq$  - (left (right)  $P$ ,  $P$ -, complete if every non decreasing sequence left (right)  $P$ -,  $P$ -(respectively) Cauchy sequence) in  $S(u)$  converges.

Through out this paper, let  $\phi: X \times X \rightarrow (-\infty, \infty]$  be a function such that

$$4. \quad \phi(x, z) \leq \phi(x, y) + \phi(y, z) \text{ for any } x, y, z \in X$$

5.  $\varphi(x, \cdot) : X \rightarrow (-\infty, \infty]$  is lower semi continuous for any  $x \in X$ ; and
6. there exists an  $x_0 \in X$  such that  $\inf_{y \in X} \varphi(x_0, y) > -\infty$

**Theorem 1.** Let  $(X, P)$  be a quasi gauge  $T_0$  space. Let  $\omega : X \times X \rightarrow [0, \infty)$  be a W-distance on  $X$  and  $\varphi : X \times X \rightarrow (-\infty, \infty]$  a function satisfying condition (4) – (6). Define a quasi-order ' $\preceq$ ' on  $X$  by  $x \preceq y$  iff  $x = y$  or  $\varphi(x, y) + \omega(x, y) \leq 0$ . Suppose that there exist a  $u \in X$  such that  $\inf_{y \in X} \varphi(u, y) > -\infty$  and  $S(u) = \{y \in X / u \preceq y\}$  is left (right)  $\leq P$ -complete.

Then the following equivalent statements hold

- (i) There exists a maximal point  $v \in S(u)$  that is

$$\forall w \in X \setminus \{v\}, \varphi(v, w) + \omega(v, w) > 0$$

- (ii) If  $T : S(u) \rightarrow 2^X$  satisfies the condition  $\forall x \in S(u) \setminus T(x)$  there exists  $y \in X \setminus \{x\}$  such that  $x \preceq y$  then  $T$  has a fixed point  $v \in S(u)$ ; that is  $v \in T(v)$ .

- (iii) A function  $f : S(u) \rightarrow X$  satisfying  $x \preceq f(x)$  for all  $x \in S(u)$  has a fixed point.

- (iv) If  $T : S(u) \rightarrow 2^{X \setminus \{\emptyset\}}$  satisfies the condition

$$\forall x \in S(u), \forall y \in T(x), x \preceq y \text{ holds}$$

then  $T$  has a stationary point  $v \in S(u)$  that is  $T(v) = \{v\}$ .

- (v) A family  $F$  of functions  $f : S(u) \rightarrow X$  satisfying  $x \preceq f(x)$  for all  $x \in S(u)$  has a common fixed point  $v \in S(u)$

- (vi) If  $Y$  is subset of  $X$  such that for each  $x \in S(u) \setminus Y$  there exists a  $z \in S(x) \setminus \{x\}$  then there exists a  $v \in S(u) \cap Y$ .

- (vii) If for each  $v \in S(u)$  with  $\inf_{y \in X} \varphi(v, y) < 0$  there exists a  $w \in S(v) \setminus \{v\}$  then there exists an  $x_0 \in S(u)$  such that  $\inf_{y \in X} \varphi(x_0, y) \leq 0$ . Any of (i)–(vi) implies (vii) conversely (vii) implies any of (i) – (vi) whenever either (a)  $\omega(x, y) = 0$  implies  $x = y$  or

$$(b) \varphi(x, x) = 0 \text{ for all } x \in X$$

**Proof.** By conditions (i) and (iv) ' $\preceq$ ' is a quasi order. Now construct inductively a sequence of points  $v_n \in S(u)$ . To each  $v_n$  we let

$$S_n = \{v \in S(u) \setminus v = v_n \text{ or } \varphi(v_n, v) + \omega(v_n, v) \leq 0\} = S(v_n)$$

And define the number

$$\gamma_n := \inf_{v \in S_n} \varphi(v_n, v)$$

Since  $\phi(v_n, \cdot)$  and  $\omega(v_n, \cdot)$  are lower semi continuous, each  $S_n$  is a closed subset of the  $\Leftarrow$  - complete subset  $S(u)$ .

$v_n \in S_n \neq \emptyset$  and that  $\gamma_n \leq 0$ . Let  $u = v_0$  then,  $S(u) = S_0$  and by the hypothesis  $\gamma_0 > \inf_{v \in X} \varphi(v_0, v) > -\infty$ . Let  $n \geq 1$  and assume that  $v_{n-1}$  with  $\gamma_{n-1} > -\infty$  is already known – that is the result is true for  $n$ . Choose  $v_n \in S_{n-1}$ , such that  $\varphi(v_{n-1}, v_n) \leq \gamma_{n-1} + 1/n$

Since  $v_n \in S_{n-1}$ , for any  $v \in S_n \setminus \{v_n\}$  we have

$$\begin{aligned} \varphi(v_{n-1}, v) + \omega(v_{n-1}, v_n) &\leq \varphi(v_{n-1}, v_n) + \omega(v_{n-1}, v_n) + \varphi(v_n, v) + \omega(v_n, v) \\ &\leq \varphi(v_n, v) + \omega(v_n, v) \leq 0 \end{aligned}$$

And hence,  $v \in S_{n-1}$ ; that is  $S_{n-1} \supset S_n$ . Therefore we obtain

$$\begin{aligned} \gamma_n = \inf_{v \in S_n} \varphi(v_n, v) &\geq \inf_{v \in S_n} \varphi(v_{n-1}, v) - \varphi(v_{n-1}, v_n) \\ &\geq \inf_{v \in S_{n-1}} (\varphi(v_{n-1}, v) - \varphi(v_{n-1}, v_n)) \\ &= \gamma_{n-1} - \varphi(v_{n-1}, v_n) \geq -1/n \end{aligned}$$

Therefore for  $v \in S_n \setminus \{v_n\}$ , we have

$$\omega(v_n, v) \leq \varphi(v_n, v) \leq -\gamma_n \leq 1/n$$

Since  $\omega$  is  $W$ -distance, for any  $\varepsilon > 0$  we can choose a sufficiently large  $n$  such that  $\omega(v_n, v) \leq 1/n$  and  $\omega(v_n, v^1) \leq 1/n$  imply  $p(v, v^1) < \varepsilon \forall p \in P$  for all  $v, v^1 \in S_n$ .

That is the diameters of the sets  $S_n$  tend to zero. Moreover for all  $k \geq n$  we have  $v_k \in S_k \subset S_n$  and hence  $p(v_n, v_k) \leq 1/n$ ,  $p(v_k, v_n) \leq 1/n$ . Thus the sequence  $\{v_n\}$  is both left and right  $P$ -Cauchy sequence in  $\Leftarrow$  left (right)  $P$ -complete set  $S(u)$  and hence converges to

some  $v^* \in S(u)$ . Clearly we have  $v^* \in \bigcap_{n=0}^{\infty} S_n$ . Since  $\dim S_n \rightarrow 0$  we have  $\bigcap_{n=0}^{\infty} S_n = \{v^*\}$  being  $X$  is a  $T_0$  space. We claim  $v^*$  is maximal; if not there exists  $w \in X \setminus \{v^*\}$  such that

$$\varphi(v^*, w) + \omega(v^*, w) \leq 0$$

Since  $v^* \in S_n \forall n$  and hence,

$$\varphi(v_n, v^*) + \omega(v_n, v^*) \leq 0$$

Then  $\varphi(v_n, w) + \omega(v_n, w) \leq 0$  by the triangular inequality.

Therefore  $w \in S_n \forall n$ . Hence  $w = v^*$  which is a contradiction. Hence

$$\varphi(v^*, w) + \omega(v^*, w) > 0 \quad \forall w \in X \setminus \{v\}.$$

(i)  $\Rightarrow$  (ii). Suppose  $v \notin T(v)$ . Then there exists a  $y \in X \setminus \{v\}$  such that

$$\varphi(v, y) + \omega(v, y) \leq 0 \text{ which contradicts (i).}$$

(ii)  $\Rightarrow$  (iii). By taking  $T = f$  as single valued map from  $S(u)$  to  $X$  then we get the following result (iii), A function  $f : S(u) \rightarrow X$  satisfying  $x \leq f(x) \quad \forall x \in S(u)$  has a fixed point.

(iii)  $\Rightarrow$  (iv). Suppose  $T$  has no stationary point. That is  $T(x) \setminus \{x\} \neq \emptyset, \forall x \in S(u)$ . Choose a function  $f$  on  $\{T(x) \setminus \{x\} \mid x \in S(u)\}$ . Then  $f$  has no fixed point by its definition. However for any  $x \in S(u)$  we have  $x \neq f(x)$  and there exists a

$y = f(x) \in T(x) \setminus \{x\}$  such that  $x \leq y = f(x)$ . Therefore by (iii),  $f$  has a fixed point, a contradiction.

(iv)  $\Rightarrow$  (v). Define a function  $T : S(u) \rightarrow 2^X$  by  $T(x) := \{f(x) \mid f \in F\}$  for all  $x \in S(u)$ . Since  $x \leq f(x)$  for all  $x \in S(u)$  and all  $f \in F$ , by (iv),  $T$  has a stationary point  $v \in S(u)$ , which is common fixed point of  $F$ .

(v)  $\Rightarrow$  (i). Suppose that for any  $x \in S(u)$ , there exists a  $y \in X \setminus \{x\}$  such that  $x \leq y$ . Choose  $f(x)$  to be one of such  $y$ . Then  $f : S(u) \rightarrow X$  has no fixed point by its definition. However  $x \leq f(x) \quad \forall x \in S(u)$ . Let  $F = \{f\}$ . By (v)  $f$  has a fixed point, which is a contradiction.

(i)  $\Rightarrow$  (vi). There exists a  $v \in S(u)$  such that  $\forall w \neq v, \varphi(v, w) + \omega(v, w) > 0$

Then by, hypothesis, we have  $v \in Y$ . Therefore  $v \in S(u) \cap Y$ .

(vi)  $\Rightarrow$  (i) for all  $x \in X$  let

$$A(x) = \{y \in X \mid x \neq y, \varphi(x, y) + \omega(x, y) \leq 0\} = S(x) \setminus \{x\}$$

Choose  $Y = \{x \in X \mid A(x) = \emptyset\}$ . If  $x \notin Y$ , then there exists a  $z \in A(x)$ . Hence by (vi) there exists  $v \in S(u) \cap Y$ . Hence  $A(v) = \emptyset$ , that is,  $\varphi(v, w) + \omega(v, w) > 0 \quad \forall w \neq v$ . Hence (i) hold.

(i)  $\Rightarrow$  (vii). Suppose that for any  $x \in S(u)$ , we have  $\inf_{y \in X} \varphi(x, y) < 0$ . Then there exists  $z \in S(x) \setminus \{x\}$ ; that is  $x \prec z$ . But by (i) there exists a maximal element  $v \in S(u)$ , that is  $v \not\prec w$  for all  $w \in X \setminus \{v\}$ , which is a contradiction.

(vii)  $\Rightarrow$  (i). Suppose that for each  $v \in S(u)$ , there exists a  $w \in X \setminus \{v\}$  such that  $\varphi(v, w) + \omega(v, w) \leq 0$  or  $w \in S(v) \setminus \{v\}$ .

Then by (vii) there exists an  $x_0 \in S(u)$  such that  $\inf_{y \in X} \varphi(x_0, y) \geq 0$ . For  $x_0 = v$  and  $x_1 = w$  we have,

$$\varphi(x_0, x_1) + \omega(x_0, x_1) \leq 0 \text{ or } x_1 \in S(x_0) \setminus \{x_0\}$$

Note that we should have  $\inf_{y \in X} \varphi(x_0, y) = 0$ ; otherwise we have a contradiction.

$$0 < \inf_{y \in X} \varphi(x_0, y) + \omega(x_0, x_1) \leq 0.$$

Since  $\omega(x_0, x_1) \geq 0$  and  $\varphi(x_0, x_1) \leq 0$ , we should have  $\varphi(x_0, x_1) = \inf_{y \in X} \varphi(x_0, y) = 0$  and hence  $\omega(x_0, x_1) = 0$

(a)  $\omega(x_0, x_1) = 0 \Rightarrow x_0 = x_1$  then we have a contradiction

(b) Similarly for  $x_1 = v \in S(u)$  and  $x_2 = w$  we have

$$\varphi(x_1, x_2) + \omega(x_1, x_2) \leq 0 \text{ or } x_2 \in S(x_1) \setminus \{x_1\}$$

Since  $x_2 \in S(x) \setminus \{x\} \subset S(x_0)$ , we have either

$$x_2 = x_0 \text{ or } \varphi(x_0, x_2) + \omega(x_0, x_2) \leq 0.$$

If  $x_2 = x_0$  then by assumption (b), we have  $\varphi(x_0, x_2) = 0$  and  $\omega(x_0, x_2) = 0$ .

If  $\varphi(x_0, x_2) + \omega(x_0, x_2) \leq 0$ ;  $\inf_{y \in X} \varphi(x_0, y) = 0$  implies  $\varphi(x_0, x_2) = 0$  and  $\omega(x_0, x_2) = 0$  as above. Now by condition (3)

$\omega(x_0, x_1) = 0 = \omega(x_0, x_2)$  implies  $p(x_1, x_2) = p(x_2, x_1) = 0$  for all  $p \in P$  that is  $x_1 = x_2$  since  $X$  is a  $T_0$  space; which is a contradiction.

The following result is a simplified form of theorem 1.

**Theorem 2.** Let  $(X, P)$  be a left (right)  $P$ -sequentially complete quasi gauge  $T_0$  space. Let  $\omega: X \times X \rightarrow [0, \infty)$  be a  $W$ -distance on  $X$  and  $\varphi: X \times X \rightarrow (-\infty, \infty]$  a function satisfying condition (4) – (6) and  $x_0 \in X$  satisfying

$\inf_{y \in X} \varphi(x_0, y) > -\infty$ , in condition (6). Define a quasi-order  $\preccurlyeq$  on  $X$  by

$$x \preccurlyeq y \text{ iff } x = y \text{ or } \varphi(x, y) + \omega(x, y) \leq 0$$



Then the following equivalent statements hold

- (i) There exists a maximal point  $v \in S(x_0)$  that is  $v \preccurlyeq w$  implies  $v = w$
- (ii) If  $T : X \rightarrow 2^X$  satisfies the condition  
 $\forall x \in X \setminus T(x)$  there exists  $y \in X \setminus \{x\}$  such that  $x \preccurlyeq y$   
then  $T$  has a fixed point.
- (iii) A function  $f : X \rightarrow X$  satisfying  $x \preccurlyeq f(x)$  for all  $x \in X$  has a fixed point.
- (iv) If  $T : X \rightarrow 2^{X \setminus \{\emptyset\}}$  satisfies the condition  
 $\forall x \in X, \forall y \in T(x), x \preccurlyeq y$   
then  $T$  has a stationary point.
- (v) A family  $F$  of functions  $f : X \rightarrow X$  such that  $x \preccurlyeq f(x)$  for all  $x \in X$  has a common fixed point.
- (vi) If  $Y$  subset of  $X$  such that for each  $x \in X \setminus Y$  there exists a  
 $z \in X \setminus \{x\}$  such that  $x \preccurlyeq z$ , then we have  $S(x_0) \cap Y \neq \emptyset$
- (vii) If, for each  $v \in X$  with  $\inf_{y \in X} \varphi(v, y) < 0$  there exists a  $w \neq v$  such that  $v \leq w$ ,  
then there exists an  $x_0 \in X$  such that  $\inf_{y \in X} \varphi(x_0, y) \geq 0$

**Proof.** Since there exists an  $x_0 \in X$  such that  $\inf_{y \in X} \varphi(x_0, y) > -\infty$  by (6), put  $u = x_0$  and  $S(u) = \{y \in X \mid u \leq y\} = \{y \in X \mid y = u \text{ or } \varphi(u, y) + \omega(u, y) \leq 0\}$ . Since  $\varphi(u, \cdot) + \omega(u, \cdot)$  is lower semi continuous by conditions (2) and (5),  $S(u)$  is a closed subset of the sequentially complete quasi gauge space  $(X, P)$ . Therefore,  $S(u)$  is sequentially complete and hence  $\preccurlyeq$  complete. Now from Theorem 1 we get the results.

Note that the result follows if we replace the function  $\omega$  by a family of  $W$ -distance  $\{\omega_p\}$  for the corresponding family of quasi-pseudo metrics  $\{p\}$  with the quasi order relation ' $\preccurlyeq$ ' on  $X$  by  $x \preccurlyeq y$  iff  $x = y$  or

$$\varphi(x, y) + \omega_p(x, y) \leq 0 \text{ for each } p \text{ in } P.$$

For  $\omega = d$  and  $\varphi(x, y) = F(y) - F(x)$  where  $F : X \rightarrow (-\infty, \infty]$  is a proper lower semi continuous function on the complete metric space  $X$  bounded from below. We obtain the results due to Ekeland [4], Phelps [17], Tuy [20], Penot [16], Caristi [2], Maschler and Peleg [10], Kasahara [9], Takahashi [19], from Theorem 2. Further in the above mentioned papers, they applied the results to various problems.

By taking  $\omega_p(x, y) = \varepsilon\lambda^{-1}p(x, y)$  and  $\phi(x, y) = F(y) - F(x)$  for every  $x, y \in X$ , we deduce the well known central result to Ekeland [4] on the variational principle for approximate solution of minimization problems from the corresponding theorem 2 for P-sequentially complete Hausdorff gauge space as given in [7].

**Theorem 3.** (Ekeland [4]). Let  $(X, P)$  be a P-sequentially complete Hausdorff gauge space and  $F : X \rightarrow (-\infty, \infty]$ , a proper lower semi continuous function bounded from below: Let  $\varepsilon > 0$  be given, and a point  $u \in X$  such that  $F(u) \leq \inf_X F + \varepsilon$ . Then for any  $\lambda > 0$ , there exists a point  $v \in B_p(u, \lambda) = \{x \in X \mid p(u, x) \leq \lambda\}$  such that,  $F(v) \leq F(u)$  and  $F(w) > F(v) - \varepsilon\lambda^{-1}p(v, w)$  for any  $w \in X$   $w \neq v$ ,  $\forall p \in P$ .

**Proof.** Let  $\omega_p(x, y) = \varepsilon\lambda^{-1}p(x, y)$  and  $\phi(x, y) = F(y) - F(x)$  for  $x, y \in X$  (here we let  $\phi(x, y) = 0$  if  $F(x) = F(y) = \infty$ ). Then conditions (1) – (5) for  $\omega_p$  and  $\phi$  are clearly satisfied. From (6), let  $x_0 = u$ , then  $\inf_{y \in X} \phi(x, y) = \inf_X F - F(x) > -\varepsilon > -\infty$  by hypothesis. By applying in theorem 2 (i) there exists a point  $v \in S(u)$  such that  $\phi(v, w) + \omega_p(v, w) > 0$  for all  $w \in X \setminus \{v\}$ . It is enough to show that  $F(u) \leq F(v)$  and  $v \in B(u, \lambda)$  since  $v \in S(u)$ , we have  $\phi(u, v) + \omega_p(u, v) \leq 0$ . Since  $\omega_p(u, v) \geq 0$  for all  $p$  in  $P$ , we should have  $\phi(u, v) = F(v) - F(u) \leq 0$  or  $F(v) \leq F(u)$ . Moreover  $\omega_p(u, v) = \varepsilon\lambda^{-1}p(u, v) \leq -\phi(u, v) = F(u) - F(v) \leq F(x_0) - \inf_X F \leq \varepsilon$  by hypothesis, and hence  $\varepsilon\lambda^{-1}p(u, v) \leq \varepsilon$  or  $p(u, v) \leq \lambda \forall p \in P$ . That is  $v \in B_p(u, \lambda) \forall p \in P$ .

However the following example shows that the result does not follow for a left (right) P-sequentially complete quasi gauge  $T_0$  space.

**Example 1.** Let  $X = [0, 1]$  with the quasi pseudo metric defined by

$$\begin{aligned} & 0 \text{ if } x = y \\ & p(x, y) = x \text{ If } x < y \\ & 1 \text{ If } x > y \end{aligned}$$

Topology is generated by the sub basic family  $\{\{x\}_{x \neq 0}, [x, 1]_{x \neq 0}, [0, 1]\}$  is a  $T_0$  space and every sequence converges to 0.  $X$  is P-sequentially complete  $\psi : X \rightarrow (-\infty, \infty]$  defined by

$$x \text{ if } x \leq \frac{1}{2},$$

$\psi(x) =$

$1 - x$  if  $\frac{1}{2} < x \leq 1$  then

$\psi(0) = \psi(1) = \inf \psi(x) = 0 \leq \inf \psi(x) + \varepsilon$ ; for any  $\varepsilon > 0$ ;  $u = 1 \in X$  such that,

$\psi(1) \leq \inf \psi(x) + \varepsilon$  then  $v = 0 \in B(u, \lambda)$  is the only point which satisfies

$\psi(v) \leq \psi(u)$  but  $\psi(v) < \psi(u) + \varepsilon \lambda^{-1} p(v, w)$ ,  $w \neq v$  does not hold for  $w = u$ .

In (3) Park obtained the above result for quasi metric space by putting  $\omega_p(x, y) = \varepsilon \lambda^{-1} p(x, y)$  in the quasi metric space version of theorem 1. But it is interesting to note that  $\omega_p(x, y)$  may not satisfy the condition (iii), even for a quasi metric space. Following example illustrate this fact.

**Example 2.** Let  $X = [0, 1]$  with the quasi metric defined by

$0$  if  $x = y$

$p(x, y) = y$  If  $x < y$

$1$  If  $x > y$

Topology is generated by the sub basic family  $\{\{x\}_{x \neq 0}, [x, \varepsilon]\}$ .  $X$  is a left  $P$ -sequentially complete quasi metric space.  $\omega_p(x, y) = \varepsilon \lambda^{-1} p(x, y)$  does not satisfy the condition (iii) for any  $0 < \varepsilon < 1$ , choose  $z = 0$  and  $x, y$  such that  $x \neq y$  and  $0 > \omega_p(z, x) = \varepsilon \lambda^{-1} p(z, x) < \delta$ ;  $0 < \omega_p(z, y) = \varepsilon \lambda^{-1} p(z, y) < \delta$  then either  $p(x, y) = 1$  or  $p(y, x) = 1$  i.e. there exist no  $\delta > 0$  such that whenever  $\omega_p(z, x) < \delta$ ;  $\omega_p(z, y) < \delta$  implies  $p(x, y) < \varepsilon$ .

But the weak form of Ekeland's variational principle in a quasi gauge setting is given in author's thesis in chapter 2 theorem 4.6 and few applications and examples were also given.

**Theorem 4.** Let  $(X, P)$  be a right  $P$ -sequentially complete quasi gauge space  $T_0$  and  $F : X \rightarrow (-\infty, \infty]$  a proper lower semi continuous function bounded from below:

Then given any  $\varepsilon > 0$ , there exists  $u_\varepsilon \in X$  such that

$F(u_\varepsilon) \leq \inf_X F + \varepsilon$  and

$F(u_\varepsilon) < F(v) + \varepsilon p(u, u_\varepsilon)$  for all  $u \in X$ , with

$u \neq u_\varepsilon$  for each  $p \in P$  (with  $p(u, u_\varepsilon) \neq 0$ ).

As a simple application of theorem 2 (iii) we extend the fixed point theorem of Park [13] which is a generalization of theorem due to Downing and Kirk [3].

**Theorem 5.** Let  $(X, P_1)$  and  $(Z, P_2)$  be left (right)  $P$ -sequentially complete quasi gauge  $T_0$  space with  $W$ -distance.  $\omega_x$  and  $\omega_z$  respectively and  $f: X \rightarrow X$  a function. Suppose there exist a closed map  $g: X \rightarrow Z$  and the function

$\phi: g(X) \times g(X) \rightarrow (-\infty, \infty]$  satisfying the condition (4) – (6) on  $g(X)$  such that

$$\phi(g(x), g(f(x))) + \max\{\omega_x(x, f(x)), \omega_z(g(x), g(f(x)))\} \leq 0 \text{ for } x \in X$$

Then  $f$  has a fixed point.

**Proof.** Define  $\omega(x, y) = \max\{\omega_x(x, y), \omega_z(g(x), g(y))\}$  for  $x, y \in X$  is a  $W$ -distance on  $X$  and the function  $\phi'$  defined on  $X \times X$  by  $\phi'(x, y) = \phi(g(x), g(y))$  for  $x, y \in X$  and satisfies conditions (4) – (6).

Hence for each  $x \in X$

$$\phi'(x, f(x)) + \omega(x, f(x)) \leq 0 \text{ or } x \preceq f(x).$$

Therefore  $f$  has a fixed point by Theorem 2(iii).

By taking  $\omega_x$  and  $\omega_z$  as the distances on metric spaces  $X$  and  $Z$  respectively and  $\phi(x, y) = F(y) - F(x)$  where  $F: X \rightarrow (-\infty, \infty]$  is a proper lower semi continuous function bounded from below. The above result reduces to the theorem due to Downing and Kirk [3].

Quasi-metric versions of Theorem 1 (i) and (iii) were due to Hicks [5]. Quasi-gauge version of Caristi's theorem is given in Jessy Antony and P V Subramaniyam [6]. Further applications of Ekeland's Variational principle and the extension to quasi-gauge space are given in author's thesis [7]. For more new applications of Theorem 1 & 2 see [8, 9, 11, 12].

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# NATURAL BOUNDARY CONDITIONS ASSOCIATED WITH THIRD ORDER LINEAR HOMOGENEOUS ORDINARY DIFFERENTIAL EQUATIONS

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**ABSTRACT :** In the paper entitled “On the Boundary Conditions Associated with Second Order Linear Homogeneous Differential Equations” [1], J. Das has established the condition for which the second order linear homogeneous Ordinary Differential Equation (ODE) may possess a non-trivial boundary condition (BC) satisfied by all solutions of the differential equation. Such a BC has been named Natural Boundary Condition (NBC). In this paper we have established a necessary and sufficient condition for the existence of NBC of third order linear homogeneous Ordinary Differential Equation. The number of linearly independent NBCs in different cases have been established along with examples.

**Key words :** Natural boundary condition.

**AMS Subject Classification (2000) :** 34B

## 1. INTRODUCTION

In the paper entitled “On the Boundary Conditions Associated with Second Order Linear Homogeneous Differential Equations” [1], J. Das has established the condition for which the second order linear homogeneous ordinary differential equation

$$p_0(t)y^{(2)}(t) + p_1(t)y^{(1)}(t) + p_2(t)y(t) = 0. \quad (1.1)$$

where  $p_0, p_1, p_2$  are continuous complex functions and  $p_0(t) \neq 0$  for all  $t \in [a, b]$  may possess a non-trivial boundary condition (BC) satisfied by all solutions of the differential equation. Such a BC has been named as Natural Boundary Condition (NBC). This idea of NBC has originated from the observation that all solutions of the differential equation  $y^{(2)}(t) + y(t) = 0$  satisfy the BCs  $y(0) + y(\pi) = 0$ ,  $y^{(1)}(0) + y^{(1)}(\pi) = 0$ . The notion of NBC can be extended to differential equations of third order

$$p_0(t)y^{(3)}(t) + p_1(t)y^{(2)}(t) + p_2(t)y^{(1)}(t) + p_3(t)y(t) = 0 \quad (1.2)$$

where  $p_0, p_1, p_2, p_3$  are continuous complex functions and  $p_0(t) \neq 0$  for all  $t \in [a, b]$ . §2 deals with some necessary preliminaries. In §3 we have established a necessary and sufficient condition for the existence of a NBC of the type

$$U_\alpha[y] = \alpha_1 y(a) + \alpha_2 y'(a) + \alpha_3 y''(a) + \alpha_4 y(b) + \alpha_5 y'(b) + \alpha_6 y''(b) = 0 \quad (1.3)$$

associated with (1.2).

The number of linearly independent NBCs associated with (1.2) in different cases has been established in §4. In §5 we have discussed some examples. The idea of compatibility for BCs will be dealt in a later paper.

## 2. NECESSARY PRELIMINARIES

Let  $\xi, \eta, \psi : [a, b] \rightarrow \mathbb{C}$  denote the linearly independent solutions of (1.2) which satisfy the initial conditions:

$$\xi(a) = 1 \quad \xi^{(1)}(a) = 0 \quad \xi^{(2)}(a) = 0 \quad (2.1)$$

$$\eta(a) = 0 \quad \eta^{(1)}(a) = 1 \quad \eta^{(2)}(a) = 0 \quad (2.2)$$

$$\psi(a) = 0 \quad \psi^{(1)}(a) = 0 \quad \psi^{(2)}(a) = 1 \quad (2.3)$$

As the coefficients  $p_0, p_1, p_2, p_3$  in (1.2) are complex-valued, the solutions  $\xi, \eta, \psi$  are also complex-valued.

$$\text{Let, } \xi(b) = \xi_1 + i\xi_2 \quad \xi^{(1)}(b) = \xi'_1 + i\xi'_2 \quad \xi^{(2)}(b) = \xi''_1 + i\xi''_2 \quad (2.4)$$

$$\eta(b) = \eta_1 + i\eta_2 \quad \eta^{(1)}(b) = \eta'_1 + i\eta'_2 \quad \eta^{(2)}(b) = \eta''_1 + i\eta''_2 \quad (2.5)$$

$$\psi(b) = \psi_1 + i\psi_2 \quad \psi^{(1)}(b) = \psi'_1 + i\psi'_2 \quad \psi^{(2)}(b) = \psi''_1 + i\psi''_2 \quad (2.6)$$

where  $\xi_r, \eta_r, \psi_r, \xi'_r, \eta'_r, \psi'_r, \xi''_r, \eta''_r, \psi''_r, (i = 1, 2)$  are real numbers.

We further note that

$$\begin{aligned} U_\alpha[\xi] &= \alpha_1 \xi(a) + \alpha_2 \xi^{(1)}(a) + \alpha_3 \xi^{(2)}(a) + \alpha_4 \xi(b) + \alpha_5 \xi^{(1)}(b) + \alpha_6 \xi^{(2)}(b) \\ &= \alpha_1 + \alpha_4 [\xi_1 + i\xi_2] + \alpha_5 [\xi'_1 + i\xi'_2] + \alpha_6 [\xi''_1 + i\xi''_2] \\ &= (\alpha_1 + \alpha_4 \xi_1 + \alpha_5 \xi'_1 + \alpha_6 \xi''_1) + i(\alpha_4 \xi_2 + \alpha_5 \xi'_2 + \alpha_6 \xi''_2) \end{aligned} \quad (2.7)$$

Similarly,

$$U_\alpha[\eta] = (\alpha_2 + \alpha_4 \eta_1 + \alpha_5 \eta'_1 + \alpha_6 \eta''_1) + i(\alpha_4 \eta_2 + \alpha_5 \eta'_2 + \alpha_6 \eta''_2) \quad (2.8)$$

$$U_\alpha[\psi] = (\alpha_3 + \alpha_4 \psi_1 + \alpha_5 \psi'_1 + \alpha_6 \psi''_1) + i(\alpha_4 \psi_2 + \alpha_5 \psi'_2 + \alpha_6 \psi''_2) \quad (2.9)$$

as  $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$  are real numbers.



## 3. THE MAIN THEOREM

**Theorem :** The DE (1.2) possesses a NBC iff

$$\Delta = \begin{vmatrix} \xi_2 & \xi'_2 & \xi''_2 \\ \eta_2 & \eta'_2 & \eta''_2 \\ \psi_2 & \psi'_2 & \psi''_2 \end{vmatrix} = 0 \quad (3.1)$$

**Proof.** Let the DE (1.2) possess a NBC,  $U_\alpha[y] = 0$  for some

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \neq (0, 0, 0, 0, 0, 0).$$

$$\text{Then } U_\alpha[\xi] = U_\alpha[\eta] = U_\alpha[\psi] = 0 \quad (3.2)$$

From (2.7) – (2.9) these imply that the following algebraic equations in  $\alpha$  have a non-trivial solution

$$\alpha_1 + \alpha_4 \xi_1 + \alpha_5 \xi'_1 + \alpha_6 \xi''_1 = 0 \quad (3.3)$$

$$\alpha_4 \xi_2 + \alpha_5 \xi'_2 + \alpha_6 \xi''_2 = 0 \quad (3.4)$$

$$\alpha_2 + \alpha_4 \eta_1 + \alpha_5 \eta'_1 + \alpha_6 \eta''_1 = 0 \quad (3.5)$$

$$\alpha_4 \eta_2 + \alpha_5 \eta'_2 + \alpha_6 \eta''_2 = 0 \quad (3.6)$$

$$\alpha_3 + \alpha_4 \psi_1 + \alpha_5 \psi'_1 + \alpha_6 \psi''_1 = 0 \quad (3.7)$$

$$\alpha_4 \psi_2 + \alpha_5 \psi'_2 + \alpha_6 \psi''_2 = 0 \quad (3.8)$$

We note from (3.3), (3.5) and (3.7) that  $(\alpha_4, \alpha_5, \alpha_6) = (0, 0, 0)$  imply  $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$ . But  $\alpha$  is non-trivial. Hence we must have  $(\alpha_4, \alpha_5, \alpha_6) \neq (0, 0, 0)$ .

This means

$$\Delta = \begin{vmatrix} \xi_2 & \xi'_2 & \xi''_2 \\ \eta_2 & \eta'_2 & \eta''_2 \\ \psi_2 & \psi'_2 & \psi''_2 \end{vmatrix} = 0 \text{ from (3.4), (3.6) and (3.8).}$$

Conversely, if  $\Delta = 0$ , we have from (3.4), (3.6) and (3.8) at least one solution  $(\alpha_4, \alpha_5, \alpha_6) \neq (0, 0, 0)$ . This  $(\alpha_4, \alpha_5, \alpha_6)$  will determine  $(\alpha_1, \alpha_2, \alpha_3)$  from (3.3), (3.5) and (3.7). In other words there is at least one NBC with respect to the DE (1.2).

#### 4. THE NUMBER OF LINEARLY INDEPENDENT NBCS IN DIFFERENT CASES

To find the number of linearly independent NBC associated with (1.2) we deal with the following cases

**Case (i).** Let  $\Delta = 0$  but at least one minor of  $\Delta$ , say  $\xi''_2 \eta'_2 - \xi'_2 \eta''_2 \neq 0$ . Here rank of matrix of  $\Delta$  is 2. Hence the set of unknowns  $(\alpha_4, \alpha_5, \alpha_6)$  form a linear space of dimension one. Hence there is one NBC as calculated below.

From (3.4) and (3.6), solving for  $\alpha_5$  and  $\alpha_6$ , we have

$$\alpha_5 = \frac{\alpha_4(\xi''_2 \eta_2 - \eta''_2 \xi_2)}{(\xi'_2 \eta''_2 - \eta'_2 \xi''_2)} \quad (4.1)$$

$$\alpha_6 = \frac{\alpha_4(\xi'_2 \eta_2 - \xi_2 \eta'_2)}{(\xi''_2 \eta'_2 - \eta''_2 \xi'_2)} \quad (4.2)$$

Substituting (4.1) and (4.2) in (3.3), (3.5) and (3.7), we have

$$\alpha_1 = \alpha_4 \left( -\xi_1 + \frac{(\xi''_2 \eta_2 - \xi_2 \eta''_2) \xi'_1}{\xi''_2 \eta'_2 - \xi'_2 \eta''_2} - \frac{(\xi'_2 \eta_2 - \xi_2 \eta'_2) \xi''_1}{\xi''_2 \eta'_2 - \xi'_2 \eta''_2} \right) \quad (4.3)$$

$$\alpha_2 = \alpha_4 \left( -\eta_1 + \frac{(\xi''_2 \eta_2 - \xi_2 \eta''_2) \eta'_1}{\xi''_2 \eta'_2 - \xi'_2 \eta''_2} - \frac{(\xi'_2 \eta_2 - \xi_2 \eta'_2) \eta''_1}{\xi''_2 \eta'_2 - \xi'_2 \eta''_2} \right) \quad (4.4)$$

$$\alpha_3 = \alpha_4 \left( -\psi_1 + \frac{(\xi''_2 \eta_2 - \xi_2 \eta''_2) \psi'_1}{\xi''_2 \eta'_2 - \xi'_2 \eta''_2} - \frac{(\xi'_2 \eta_2 - \xi_2 \eta'_2) \psi''_1}{\xi''_2 \eta'_2 - \xi'_2 \eta''_2} \right) \quad (4.5)$$

Substituting (4.1) – (4.5) in  $U_\alpha [\gamma] = 0$  we have the required NBC

$$\begin{aligned} & \{-\xi_1(\xi''_2 \eta'_2 - \xi'_2 \eta''_2) + \xi'_1(\xi''_2 \eta_2 - \eta''_2 \xi_2) - (\xi'_2 \eta_2 - \xi_2 \eta'_2) \xi''_1\} \gamma(a) \\ & + \{-\eta_1(\xi''_2 \eta'_2 - \xi'_2 \eta''_2) + \eta'_1(\xi''_2 \eta_2 - \eta''_2 \xi_2) - (\xi'_2 \eta_2 - \xi_2 \eta'_2) \eta''_1\} \gamma^{(1)}(a) \\ & + \{-\psi_1(\xi''_2 \eta'_2 - \xi'_2 \eta''_2) + \psi'_1(\xi''_2 \eta_2 - \eta''_2 \xi_2) - (\xi'_2 \eta_2 - \xi_2 \eta'_2) \psi''_1\} \gamma^{(2)}(a) \\ & + (\xi'_2 \eta''_2 - \eta'_2 \xi''_2) \gamma(b) + (\xi''_2 \eta_2 - \eta''_2 \xi_2) \gamma^{(1)}(b) - (\xi'_2 \eta_2 - \eta'_2 \xi_2) \gamma^{(2)}(b) = 0 \end{aligned} \quad (4.6)$$

If other minors of  $\Delta$  are not zero, then equivalent forms of NBCs are similarly obtained.

**Case (ii).**  $\Delta = 0$ , all minors are zero but the individual elements of  $\Delta$  are not all zero.

In this case matrix of  $\Delta$  is of rank 1. Hence the set of unknowns  $(\alpha_4, \alpha_5, \alpha_6)$  form a linear space of dimension two. We take  $(\alpha_5, \alpha_6)$  as  $(1, 0)$  and  $(0, 1)$  respectively.

Suppose  $\xi_2 \neq 0$  in  $\Delta$ , and  $(\alpha_5, \alpha_6) = (1, 0)$

$$\text{From (3.4), } \alpha_4 = -\frac{\xi'_2}{\xi_2}.$$

Substituting  $(\alpha_4, \alpha_5, \alpha_6) = \left(-\frac{\xi'_2}{\xi_2}, 1, 0\right)$  in (3.3), (3.5), (3.7) we have

$$\alpha_1 = \frac{\xi'_2 \xi_1 - \xi'_1 \xi_2}{\xi_2}$$

$$\alpha_2 = \frac{\xi'_2 \eta_1 - \xi'_1 \eta'_1}{\xi_2}$$

$$\alpha_3 = \frac{\xi'_2 \psi_1 - \xi'_1 \psi'_1}{\xi_2}$$

Putting the values of  $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$  in  $U_\alpha[y] = 0$  we have the required NBC as

$$\begin{aligned} &(\xi'_2 \xi_1 - \xi'_1 \xi_2)y(a) + (\xi'_2 \eta_1 - \xi'_1 \eta'_1)y^{(1)}(a) + (\xi'_1 \psi_1 - \xi_2 \psi'_1)y^{(2)}(a) \\ &- \xi'_2 y(b) + \xi_2 y^{(1)}(b) = 0 \end{aligned} \quad (4.6)$$

Again taking  $(\alpha_5, \alpha_6) = (0, 1)$  we similarly get the other linearly independent NBC as

$$\begin{aligned} &(\xi''_2 \xi_1 - \xi''_1 \xi_2)y(a) + (\xi''_2 \eta_1 - \xi''_1 \eta'_1)y^{(1)}(a) - (\xi''_2 \psi_1 - \xi_2 \psi''_1)y^{(2)}(a) \\ &- \xi''_2 y(b) + \xi_2 y^{(1)}(b) = 0 \end{aligned} \quad (4.7)$$

If other elements of  $\Delta$  are not zero, then equivalent forms of NBC are similarly obtained.

**Case (iii)**  $\Delta = 0$ , all elements of  $\Delta$  are zero.

In this case matrix  $\Delta$  has rank 0. Hence the set of unknowns  $(\alpha_4, \alpha_5, \alpha_6)$  form a linear space of dimension three.

Taking  $(\alpha_4, \alpha_5, \alpha_6) = (1, 0, 0)$  we obtain from (3.3), (3.5) and (3.7)

$$\alpha_1 = -\xi_1, \alpha_2 = -\eta_1 \text{ and } \alpha_3 = -\psi_1 \text{ respectively.}$$

Hence the required NBC is

$$-\xi_1 y(a) - \eta_1 y^{(1)}(a) - \psi_1 y^{(2)}(a) + y(b) = 0 \quad (4.8)$$

Similarly taking  $(\alpha_4, \alpha_5, \alpha_6) = (0, 1, 0)$  and  $(\alpha_4, \alpha_5, \alpha_6) = (0, 0, 1)$  we obtain the two other linearly independent NBC as

$$-\xi'_1 y(a) - \eta'_1 y^{(1)}(a) - \psi'_1 y^{(2)}(a) + y^{(1)}(b) = 0 \quad (4.9)$$

and 
$$-\xi''_1 y(a) - \eta''_1 y^{(1)}(a) - \psi''_1 y^{(2)}(a) + y^{(2)}(b) = 0 \quad (4.10)$$

### 5. EXAMPLES

(i) Example of Case (i) : Consider the DE

$$y^{(3)}(t) - 3iy^{(2)}(t) - 2y^{(1)}(t) = 0, \quad t \in [0, \pi] \quad (5.1)$$

Linearly independent solutions of (5.1) satisfying the initial conditions (2.1) – (2.3) are

$$\xi(t) = 1, \quad \eta(t) = \frac{i}{2}(3 + e^{2it} - 4e^{it}), \quad \psi(t) = \frac{1}{2}(1 + e^{2it} - 2e^{it})$$

$$\begin{array}{lll} \text{Here, } \xi_1 = 1, \xi_2 = 0 & \eta_1 = 0, \eta_2 = 4 & \psi_1 = -2, \psi_2 = 6 \\ \xi'_1 = 0, \xi'_2 = 0 & \eta'_1 = -3, \eta'_2 = 0 & \psi'_1 = 0, \psi'_2 = -2 \\ \xi''_1 = 0, \xi''_2 = 0 & \eta''_1 = 0, \eta''_2 = -4 & \psi''_1 = 3, \psi''_2 = 0 \end{array}$$

$$\text{Here } \Delta = 0, \text{ but the minors } \Delta_1 = \begin{vmatrix} \eta'_2 & \psi'_2 \\ \eta''_2 & \psi''_2 \end{vmatrix} = -8 \neq 0 \text{ and } \Delta_2 = \begin{vmatrix} \eta_2 & \psi_2 \\ \eta'_2 & \psi'_2 \end{vmatrix} = -8 \neq 0$$

From (3.6) and (3.8) we get  $\alpha_4 = \alpha_6$  and  $\alpha_5 = 0$

From (3.3), (3.5) and (3.7), we get  $\alpha_1 = -\alpha_4$ ,  $\alpha_2 = 0$  and  $\alpha_3 = -\alpha_4$

$$\text{Hence the required NBC is } y(0) + y^{(2)}(0) - y(\pi) - y^{(2)}(\pi) = 0 \quad (5.2)$$

(ii) Example of Case (ii): Consider the DE

$$y^{(3)}(t) - iy^{(2)}(t) = 0, \quad t \in [0, \pi] \quad (5.3)$$

Linearly independent solutions of (5.3) are

$$\xi(t) = 1, \quad \eta(t) = t, \quad \psi(t) = 1 + it - e^{it}$$

$$\begin{array}{lll}
\text{Here, } \xi_1 = 1, \xi_2 = 0 & \eta_1 = \pi, \eta_2 = 0 & \psi_1 = 2, \psi_2 = \pi \\
\xi'_1 = 0, \xi'_2 = 0 & \eta'_1 = 1, \eta'_2 = 0 & \psi'_1 = 0, \psi'_2 = 2 \\
\xi''_1 = 0, \xi''_2 = 0 & \eta''_1 = 0, \eta''_2 = 0 & \psi''_1 = -1, \psi''_2 = 0
\end{array}$$

Here  $\Delta = 0$ , all minors are zero, but  $\psi_2 = \pi, \psi'_2 = 2$

Taking  $(\alpha_5, \alpha_6) = (1, 0)$  we get from (3.8),  $\pi\alpha_4 + 2\alpha_5 = 0$

Hence,  $\alpha_4 = -2/\pi$

Substituting  $(\alpha_4, \alpha_5, \alpha_6) = \left(-\frac{2}{\pi}, 1, 0\right)$  in (3.3), (3.5), and (3.7), we have

$$\alpha_1 = \frac{2}{\pi}, \alpha_2 = 1 \text{ and } \alpha_3 = \frac{4}{\pi}$$

Hence the required NBC is  $2y(0) + \pi y^{(1)}(0) + 4y^{(2)}(0) - 2y(\pi) + \pi y^{(1)}(\pi) = 0$  (5.4)

Again putting  $(\alpha_5, \alpha_6) = (0, 1)$  in (3.8) we get  $\alpha_4 = 0$ .

Putting  $(\alpha_4, \alpha_5, \alpha_6) = (0, 0, 1)$  in (3.3), (3.5) and (3.7) we get

$$\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 1$$

Hence the required NBC is  $y^{(2)}(0) + y^{(2)}(\pi) = 0$  (5.5)

(iii) Example of Case (iii) : Consider the DE

$$y^{(3)}(t) + y^{(1)}(t) = 0, t \in [0, \pi] \quad (5.6)$$

Linearly independent solutions of (5.6) satisfying the initial conditions are

$$\xi(t) = 1, \eta(t) = \sin t, \psi(t) = 1 - \cos t$$

$$\begin{array}{lll}
\text{Here, } \xi_1 = \xi_2 = 0 & \eta_1 = 0, \eta_2 = 0 & \psi_1 = 2, \psi_2 = 0 \\
\xi'_1 = 0, \xi'_2 = 0 & \eta'_1 = -1, \eta'_2 = 0 & \psi'_1 = 0, \psi'_2 = 0 \\
\xi''_1 = 0, \xi''_2 = 0 & \eta''_1 = 0, \eta''_2 = 0 & \psi''_1 = -1, \psi''_2 = 0
\end{array}$$

Hence, the required linearly independent NBCs are (from 4.8, 4.9, 4.10).

$$y(0) + 2y^{(2)}(0) - y(\pi) = 0 \quad (5.7)$$

$$y^{(1)}(0) + y^{(1)}(\pi) = 0 \quad (5.8)$$

$$y^{(2)}(0) + y^{(2)}(\pi) = 0 \quad (5.9)$$

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# SOME COMMON UNIQUE FIXED POINT THEOREMS FOR RANDOM OPERATORS IN HILBERT SPACE

NEERAJ MALVIYA AND PANKAJ KUMAR JHADE

**ABSTRACT :** We construct a sequence of measurable functions and consider its convergence to the unique common random fixed point of a pair of non-commuting continuous random operators defined on a non empty closed subset of a separable Hilbert space.

**Key words :** Separable Hilbert space, random operators, common random fixed point.

**Mathematics Subject classification:** 47H10, 54H25.

## 1. INTRODUCTION

In recent years, the study of random fixed points have attracted much attention, some of the recent literatures in random fixed points may be noted in [1, 5]. In the present paper, we work out two common random fixed point theorems for pair of non-commuting continuous random operators defined on a non empty closed subset of a separable Hilbert space. For the purpose of obtaining the random fixed point of pair of random operators, we have used two inequalities [from 3 and 4] and the parallelogram law.

Throughout this paper,  $(\Omega, \Sigma)$  denotes a measurable space consisting of a set  $\Omega$  and sigma algebra  $\Sigma$  of subsets of  $\Omega$ ,  $H$  stands for a separable Hilbert space and  $C$  is a nonempty closed subset of  $H$ .

## 2. PRELIMINARIES

**Definition 2.1:** A function  $f : \Omega \rightarrow C$  is said to be measurable if  $f^{-1}(B \cap C) \in \Sigma$  for every Borel subset  $B$  of  $H$ .

**Definition 2.2:** A function  $F : \Omega \times C \rightarrow C$  is said to be a random operator if  $F(.,x) : \Omega \rightarrow C$  is measurable for every  $x \in C$ .

**Definition 2.3:** A measurable function  $g : \Omega \rightarrow C$  is said to be a random fixed point of the random operator  $F : \Omega \times C \rightarrow C$ , if  $F(t, g(t)) = g(t)$  for all  $t \in \Omega$ .

**Definition 2.4:** A random operator  $F : \Omega \times C \rightarrow C$  is said to be continuous if for fixed  $t \in \Omega$ ,  $F(t,.) : C \rightarrow C$  is continuous.

**Condition (A):** Two non-commuting mappings  $S, T : C \rightarrow C$  where  $C$  is a non empty subset of a Hilbert space  $H$ , is said to satisfy condition (A) if

$$\begin{aligned} \|STx - TSy\|^2 &\leq a_1\|x - STx\|^2 + a_2\|y - TSy\|^2 + a_3\|x - TSy\|^2 + a_4\|y - STx\|^2 \\ &\quad + a_5\|x - y\|^2 \end{aligned} \quad (2.1)$$

for all  $x, y$  in  $C$  and  $a_1, a_2, a_3, a_4, a_5$  being positive real numbers such that

$$(a_1 + 2a_2 + 4a_3 + a_5) < 1 \text{ and } (a_3 + a_4 + a_5) < 1 \quad (2.2)$$

**Condition (B):** Two non-commuting mappings  $S, T : C \rightarrow C$  where  $C$  is a non empty subset of a Hilbert space  $H$ , is said to satisfy condition (B) if

$$\|STx - TSy\|^2 \leq \alpha \max\{\|x - STx\|^2, \|y - TSy\|^2, \|x - TSy\|^2, \|y - STx\|^2, \|x - y\|^2\} \quad (2.3)$$

$$\text{for all } x, y \text{ in } C \text{ and } 0 < \alpha < \frac{1}{4} \quad (2.4)$$

### 3. MAIN RESULTS

**Theorem 3.1** Let  $C$  be a nonempty closed subset of a separable Hilbert space  $H$ . Let  $S$  and  $T$  be two continuous non commuting random operators defined on  $C$  such that for  $t \in \Omega$ ,  $S(t, \cdot), T(t, \cdot) : C \rightarrow C$  satisfy condition (A). Then  $ST$  and  $TS$  have a common unique random fixed point in  $C$ .

**Proof.** Let  $g_0 : \Omega \rightarrow C$  be an arbitrary measurable function and choose a function  $g_1 : \Omega \rightarrow C$  such that  $g_1(t) = ST(t, g_0(t))$  for each  $t \in \Omega$ . Again, we choose a function  $g_2 : \Omega \rightarrow C$  such that  $g_2(t) = TS(t, g_1(t))$  for each  $t \in \Omega$ . Similarly, proceeding in the same way, we get a sequence of functions  $g_n : \Omega \rightarrow C$  such that for  $n \geq 0$  and for any  $t \in \Omega$ ,

$$g_{2n+1}(t) = ST(t, g_{2n}(t)), \quad g_{2n+2}(t) = TS(t, g_{2n+1}(t)) \quad (3.1)$$

If  $g_{2n}(t) = g_{2n+1}(t) = g_{2n+2}(t)$  for  $t \in \Omega$  and for some  $n$  then we see that  $g_{2n}(t)$  is a random fixed point of  $ST$  and  $TS$ ; therefore we suppose that no two consecutive terms of sequence  $\{g_n(t)\}$  for  $t \in \Omega$ , are equal.

Now consider for  $t \in \Omega$

$$\begin{aligned} \|g_{2n+1}(t) - g_{2n+2}(t)\|^2 &= \|ST(t, g_{2n}(t)) - TS(t, g_{2n+1}(t))\|^2 \\ &\leq a_1\|g_{2n}(t) - ST(t, g_{2n}(t))\|^2 + a_2\|g_{2n+1}(t) - TS(t, g_{2n+1}(t))\|^2 \\ &\quad + a_3\|g_{2n}(t) - TS(t, g_{2n+1}(t))\|^2 + a_4\|g_{2n+1}(t) - ST(t, g_{2n}(t))\|^2 + a_5\|g_{2n}(t) - g_{2n+1}(t)\|^2 \text{ (by 2.1)} \\ &= a_1\|g_{2n}(t) - g_{2n+1}(t)\|^2 + a_2\|g_{2n+1}(t) - g_{2n+2}(t)\|^2 + a_3\|g_{2n}(t) - g_{2n+2}(t)\|^2 \\ &\quad + a_4\|g_{2n+1}(t) - g_{2n+1}(t)\|^2 + a_5\|g_{2n}(t) - g_{2n+1}(t)\|^2 \\ &\leq a_1\|g_{2n}(t) - g_{2n+1}(t)\|^2 + a_2\|g_{2n+1}(t) - g_{2n+2}(t)\|^2 + 2a_3\|g_{2n}(t) - g_{2n+1}(t)\|^2 \end{aligned}$$



$$\begin{aligned}
& + 2a_3 \|g_{2n+1}(t) - g_{2n+2}(t)\|^2 + a_5 \|g_{2n}(t) - g_{2n+1}(t)\|^2 \\
& \quad \text{[by parallelogram law } \|x + y\|^2 \leq 2\|x\|^2 + 2\|y\|^2] \\
\Rightarrow & (1 - a_2 - 2a_3) \|g_{2n+1}(t) - g_{2n+2}(t)\|^2 \leq (a_1 + 2a_3 + a_5) \|g_{2n}(t) - g_{2n+1}(t)\|^2 \\
\Rightarrow & \|g_{2n+1}(t) - g_{2n+2}(t)\| \leq \left[ \frac{(a_1 + 2a_3 + a_5)}{(1 - a_2 - 2a_3)} \right]^{\frac{1}{2}} \|g_{2n}(t) - g_{2n+1}(t)\| \\
\Rightarrow & \|g_{2n+1}(t) - g_{2n+2}(t)\| \leq k \|g_{2n}(t) - g_{2n+1}(t)\| \text{ Where } k = \left[ \frac{(a_1 + 2a_3 + a_5)}{(1 - a_2 - 2a_3)} \right]^{\frac{1}{2}} < 1 \text{ (by 2.2)}
\end{aligned}$$

*In general*

$$\begin{aligned}
& \|g_n(t) - g_{n+1}(t)\| \leq k \|g_{n-1}(t) - g_n(t)\| \\
\Rightarrow & \|g_n(t) - g_{n+1}(t)\| \leq k^n \|g_0(t) - g_1(t)\| \quad (3.2)
\end{aligned}$$

Now we shall prove that for  $t \in \Omega$ ,  $\{g_n(t)\}$  is a Cauchy sequence.

For this for every positive integer  $p$  we have,

$$\begin{aligned}
\|g_n(t) - g_{n+p}(t)\| &= \|g_n(t) - g_{n+1}(t) + g_{n+1}(t) - \dots + g_{n+p-1}(t) - g_{n+p}(t)\| \\
&\leq \|g_n(t) - g_{n+1}(t)\| + \|g_{n+1}(t) - g_{n+2}(t)\| + \dots + \|g_{n+p-1}(t) - g_{n+p}(t)\| \\
&\leq (k^n + k^{n+1} + k^{n+2} + \dots + k^{n+p-1}) \|g_0(t) - g_1(t)\| \quad (\text{by 3.2}) \\
&= k^n (1 + k + k^2 + \dots + k^{p-1}) \|g_0(t) - g_1(t)\| \\
&\leq \frac{k^n}{1-k} \|g_0(t) - g_1(t)\| \text{ for all } t \in \Omega
\end{aligned}$$

as  $n \rightarrow \infty$ ,  $\|g_n(t) - g_{n+p}(t)\| \rightarrow 0$ , it follows that for  $t \in \Omega$ ,  $\{g_n(t)\}$  is a Cauchy sequence and hence is convergent in Hilbert space  $H$ .

$$\text{For } t \in \Omega, \text{ let } \{g_n(t)\} \rightarrow g(t) \text{ as } n \rightarrow \infty, \quad (3.3)$$

Since  $C$  is closed,  $g$  is a function from  $C$  to  $C$ .

### Existence of random fixed point:

Consider

$$\begin{aligned}
\|g(t) - TS(t, g(t))\|^2 &= \|g(t) - g_{2n+1}(t) + g_{2n+1}(t) - TS(t, g(t))\|^2 \\
&\leq 2\|g(t) - g_{2n+1}(t)\|^2 + 2\|g_{2n+1}(t) - TS(t, g(t))\|^2 \\
&\quad \text{[by parallelogram law } \|x + y\|^2 \leq 2\|x\|^2 + 2\|y\|^2] \\
&= 2\|g(t) - g_{2n+1}(t)\|^2 + 2\|ST(t, g_{2n}(t)) - TS(t, g(t))\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq 2\|g(t) - g_{2n+1}(t)\|^2 + 2a_1\|g_{2n}(t) - ST(t, g_{2n}(t))\|^2 + 2a_2\|g(t) - TS(t, g(t))\|^2 \\
&+ 2a_3\|g_{2n}(t) - TS(t, g(t))\|^2 + 2a_4\|g(t) - ST(t, g_{2n}(t))\|^2 + 2a_5\|g_{2n}(t) - g(t)\|^2 \quad (\text{by 2.1}) \\
&= 2\|g(t) - g_{2n+1}(t)\|^2 + 2a_1\|g_{2n}(t) - g_{2n+1}(t)\|^2 + 2a_2\|g(t) - TS(t, g(t))\|^2 \\
&+ 2a_3\|g_{2n}(t) - TS(t, g(t))\|^2 + 2a_4\|g(t) - g_{2n+1}(t)\|^2 + 2a_5\|g_{2n}(t) - g(t)\|^2
\end{aligned}$$

As  $\{g_{2n+1}(t)\}$  and  $\{g_{2n}(t)\}$  are subsequences of  $\{g_n(t)\}$  for  $t \in \Omega$ ,

as  $n \rightarrow \infty$ ,  $\{g_{2n+1}(t)\} \rightarrow g(t)$  and  $\{g_{2n}(t)\} \rightarrow g(t)$  for all  $t \in \Omega$

$$\Rightarrow \|g(t) - TS(t, g(t))\|^2 \leq (2a_2 + 2a_3)\|g(t) - TS(t, g(t))\|^2$$

$$\Rightarrow (1 - 2a_2 - 2a_3)\|g(t) - TS(t, g(t))\|^2 \leq 0$$

$$\Rightarrow \|g(t) - TS(t, g(t))\|^2 = 0 \quad (\text{by 2.2 } (2a_2 + 2a_3) < 1)$$

$$\Rightarrow TS(t, g(t)) = g(t), \text{ for all } t \in \Omega \quad (3.4)$$

In an exactly similar way we can prove that for all  $t \in \Omega$ ,

$$ST(t, g(t)) = g(t) \quad (3.5)$$

Again, if  $A : \Omega \times C \rightarrow C$  is a continuous random operator on a non-empty subset  $C$  of a separable Hilbert space  $H$ , then for any measurable function  $f : \Omega \rightarrow C$ , the function

$h(t) = A(t, f(t))$  is also measurable [2].

It follows from the construction of  $\{g_n(t)\}$  (by 3.1) and the above consideration that  $\{g_n(t)\}$  is a sequence of measurable functions. From (3.3) it follows that  $g(t)$  for  $t \in \Omega$ , is also a measurable function. This fact along with (3.4 & 3.5) shows that  $\xi : \Omega \rightarrow C$  is a common random fixed point of  $ST$  &  $TS$ .

### Uniqueness—

Let  $h : \Omega \rightarrow C$  be another random fixed point common to  $ST$  &  $TS$ , that is,

for  $t \in \Omega$ ,

$$ST(t, h(t)) = h(t)$$

$$TS(t, h(t)) = h(t)$$

Then for  $t \in \Omega$

$$\begin{aligned}
\|g(t) - h(t)\|^2 &= \|ST(t, g(t)) - TS(t, h(t))\|^2 \\
&\leq a_1\|g(t) - ST(t, g(t))\|^2 + a_2\|h(t) - TS(t, h(t))\|^2 \\
&+ a_3\|g(t) - TS(t, h(t))\|^2 + a_4\|h(t) - ST(t, g(t))\|^2 + a_5\|g(t) - h(t)\|^2 \\
&= a_1\|g(t) - g(t)\|^2 + a_2\|h(t) - h(t)\|^2 + a_3\|g(t) - h(t)\|^2
\end{aligned}$$

$$\begin{aligned}
& + a_4 \|h(t) - g(t)\|^2 + a_5 \|g(t) - h(t)\|^2 \\
\Rightarrow & (1 - a_3 - a_4 - a_5) \|g(t) - h(t)\|^2 \leq 0 \\
\Rightarrow & \|g(t) - h(t)\|^2 = 0 \text{ [by (2.2)]} \\
\Rightarrow & g(t) = h(t) \text{ for all } t \in \Omega
\end{aligned}$$

This completes the proof of the theorem 3.1

**Theorem 3.2** Let  $C$  be a nonempty closed subset of a separable Hilbert space  $H$ . Let  $S$  and  $T$  be two continuous non commuting random operators defined on  $C$  such that for  $t \in \Omega$ ,  $S(t, \cdot)$ ,  $T(t, \cdot) : C \rightarrow C$  satisfy condition (B). Then  $ST$  and  $TS$  have a common unique random fixed point in  $C$ .

**Proof.** Let  $g_0 : \Omega \rightarrow C$  be an arbitrary measurable function and choose a function  $g_1 : \Omega \rightarrow C$  such that  $g_1(t) = ST(t, g_0(t))$  for each  $t \in \Omega$ . Again, we choose a function  $g_2 : \Omega \rightarrow C$  such that  $g_2(t) = TS(t, g_1(t))$  for each  $t \in \Omega$ . Similarly, proceeding in the same way, we get a sequence of functions  $g_n : \Omega \rightarrow C$

such that for  $n \geq 0$  and for any  $t \in \Omega$ ,

$$g_{2n+1}(t) = ST(t, g_{2n}(t)), \quad g_{2n+2}(t) = TS(t, g_{2n+1}(t)) \quad (3.6)$$

If  $g_{2n}(t) = g_{2n+1}(t) = g_{2n+2}(t)$  for  $t \in \Omega$  and for some  $n$  then we see that  $g_{2n}(t)$  is a random fixed point of  $ST$  and  $TS$  therefore we suppose that no two consecutive terms of sequence  $\{g_n(t)\}$  for  $t \in \Omega$ , are equal (3.7).

Now consider for  $t \in \Omega$

$$\begin{aligned}
\|g_{2n+1}(t) - g_{2n+2}(t)\|^2 &= \|ST(t, g_{2n}(t)) - TS(t, g_{2n+1}(t))\|^2 \\
&\leq \alpha \max\{\|g_{2n}(t) - ST(t, g_{2n}(t))\|^2, \|g_{2n+1}(t) - TS(t, g_{2n+1}(t))\|^2, \\
&\quad \|g_{2n}(t) - TS(t, g_{2n+1}(t))\|^2, \|g_{2n+1}(t) - ST(t, g_{2n}(t))\|^2, \|g_{2n}(t) - g_{2n+1}(t)\|^2\} \text{ (by 2.3)} \\
&= \alpha \max\{\|g_{2n}(t) - g_{2n+1}(t)\|^2, \|g_{2n+1}(t) - g_{2n+2}(t)\|^2, \|g_{2n}(t) - g_{2n+2}(t)\|^2, \\
&\quad \|g_{2n+1}(t) - g_{2n+1}(t)\|^2, \|g_{2n}(t) - g_{2n+1}(t)\|^2\} \\
&= \alpha \max\{\|g_{2n}(t) - g_{2n+1}(t)\|^2, \|g_{2n+1}(t) - g_{2n+2}(t)\|^2, \|g_{2n}(t) - g_{2n+2}(t)\|^2\}
\end{aligned}$$

Case I

$$\begin{aligned}
\|g_{2n+1}(t) - g_{2n+2}(t)\|^2 &\leq \alpha \|g_{2n}(t) - g_{2n+1}(t)\|^2 \\
\Rightarrow \|g_{2n+1}(t) - g_{2n+2}(t)\| &\leq \alpha^{1/2} \|g_{2n}(t) - g_{2n+1}(t)\|
\end{aligned}$$

In general

$$\|g_n(t) - g_{n+1}(t)\| \leq k \|g_{n-1}(t) - g_n(t)\| \text{ where } k = \alpha^{1/2} < \frac{1}{4} \text{ (by 2.4)}$$

$$\Rightarrow \|g_n(t) - g_{n+1}(t)\| \leq k^n \|g_0(t) - g_1(t)\| \quad (3.8)$$

Case II

$$\|g_{2n+1}(t) - g_{2n+2}(t)\|^2 \leq \alpha \|g_{2n+1}(t) - g_{2n+2}(t)\|^2$$

$$\Rightarrow (1 - \alpha) \|g_{2n+1}(t) - g_{2n+2}(t)\|^2 \leq 0$$

$$\Rightarrow \|g_{2n+1}(t) - g_{2n+2}(t)\|^2 = 0 \text{ (as } \alpha < \frac{1}{4} \text{ by 2.4)}$$

$$\Rightarrow g_{2n+1}(t) = g_{2n+2}(t) \text{ in general } g_n(t) = g_{n+1}(t)$$

Which is contradiction so by remark (3.7)  $g_{2n+1}(t)$  is fixed point of  $ST$  and  $TS$ .

Case III

$$\|g_{2n+1}(t) - g_{2n+2}(t)\|^2 \leq \alpha \|g_{2n}(t) - g_{2n+2}(t)\|^2$$

$$\Rightarrow \|g_{2n+1}(t) - g_{2n+2}(t)\|^2 \leq 2\alpha \|g_{2n}(t) - g_{2n+1}(t)\|^2 + 2\alpha \|g_{2n+1}(t) - g_{2n+2}(t)\|^2$$

$$[\text{by parallelogram law } \|x + y\|^2 \leq 2\|x\|^2 + 2\|y\|^2]$$

$$\Rightarrow (1 - 2\alpha) \|g_{2n+1}(t) - g_{2n+2}(t)\|^2 \leq 2\alpha \|g_{2n}(t) - g_{2n+1}(t)\|^2$$

$$\Rightarrow \|g_{2n+1}(t) - g_{2n+2}(t)\|^2 \leq \frac{2\alpha}{(1 - 2\alpha)} \|g_{2n}(t) - g_{2n+1}(t)\|^2$$

In general

$$\Rightarrow \|g_n(t) - g_{n+1}(t)\| \leq k \|g_{n-1}(t) - g_n(t)\| \text{ where } k = \left[ \frac{2\alpha}{(1 - 2\alpha)} \right]^{\frac{1}{2}} < 1 \text{ (by 2.4)}$$

$$\Rightarrow \|g_n(t) - g_{n+1}(t)\| \leq k^n \|g_0(t) - g_1(t)\| \quad (3.9)$$

We can prove that,  $\{g_n(t)\}$  is a Cauchy sequence (using 3.8 and 3.9) and hence is convergent in Hilbert space  $H$ . (as proved in theorem 3.1)

$$\text{For } t \in \Omega, \text{ let } \{g_n(t)\} \rightarrow g(t) \text{ as } n \rightarrow \infty, \quad (3.10)$$

Since  $C$  is closed,  $g$  is a function from  $C$  to  $C$ .

**Existence of random fixed point:**

Consider

$$\|g(t) - TS(t, g(t))\|^2 = \|g(t) - g_{2n+1}(t) + g_{2n+1}(t) - TS(t, g(t))\|^2$$

$$\leq 2\|g(t) - g_{2n+1}(t)\|^2 + 2\|g_{2n+1}(t) - TS(t, g(t))\|^2$$

$$[\text{by parallelogram law } \|x + y\|^2 \leq 2\|x\|^2 + 2\|y\|^2]$$

$$= 2\|g(t) - g_{2n+1}(t)\|^2 + 2\|ST(t, g_{2n}(t)) - TS(t, g(t))\|^2$$

$$\leq 2\|g(t) - g_{2n+1}(t)\|^2 + 2\alpha \max\{\|g_{2n}(t) - ST(t, g_{2n}(t))\|^2, \|g(t) - TS(t, g(t))\|^2, \|g_{2n}(t) - TS(t, g(t))\|^2, \|g(t) - ST(t, g_{2n}(t))\|^2, \|g_{2n}(t) - g(t)\|^2\} \quad (\text{by 2.3})$$

$$= 2\|g(t) - g_{2n+1}(t)\|^2 + 2\alpha \max\{\|g_{2n}(t) - g_{2n+1}(t)\|^2, \|g(t) - TS(t, g(t))\|^2, \|g_{2n}(t) - TS(t, g(t))\|^2, \|g(t) - g_{2n+1}(t)\|^2, \|g_{2n}(t) - g(t)\|^2\} \quad (\text{by 2.3})$$

As  $\{g_{2n+1}(t)\}$  and  $\{g_{2n}(t)\}$  are subsequences of  $\{g_n(t)\}$  for  $t \in \Omega$ ,

as  $n \rightarrow \infty$ ,  $\{g_{2n+1}(t)\} \rightarrow g(t)$ ,  $\{g_{2n}(t)\} \rightarrow g(t)$  for  $t \in \Omega$ ,

Therefore

$$\begin{aligned} \|g(t) - TS(t, g(t))\|^2 &\leq 2\alpha \|g(t) - TS(t, g(t))\|^2 \\ \Rightarrow (1 - 2\alpha) \|g(t) - TS(t, g(t))\|^2 &\leq 0 \\ \Rightarrow \|g(t) - TS(t, g(t))\|^2 &= 0 \quad (\text{as } \alpha < \frac{1}{4} \text{ by 2.3}) \\ \Rightarrow TS(t, g(t)) &= g(t) \text{ for all } t \in \Omega \end{aligned} \quad (3.11)$$

In an exactly similar way we can prove that for all  $t \in \Omega$ ,

$$ST(t, g(t)) = g(t) \quad (3.12)$$

Again, if  $A : \Omega \times C \rightarrow C$  is a continuous random operator on a non-empty subset  $C$  of a separable Hilbert space  $H$ , then for any measurable function  $f : \Omega \rightarrow C$ , the function  $h(t) = A(t, f(t))$  is also measurable [2].

It follows from the construction of  $\{g_n(t)\}$  (by 3.6) and the above consideration that  $\{g_n(t)\}$  is a sequence of measurable functions. From (3.10) it follows that  $g(t)$  for  $t \in \Omega$ , is also a measurable function. This fact along with (3.11 & 3.12) shows that  $g : \Omega \rightarrow C$  is a common random fixed point of  $ST$  &  $TS$ .

**Uniqueness:**

Let  $h : \Omega \rightarrow C$  be another random fixed point common to  $ST$  &  $TS$ , that is,

for  $t \in \Omega$ ,

$$ST(t, h(t)) = h(t)$$

$$TS(t, h(t)) = h(t)$$

Then for  $t \in \Omega$

$$\begin{aligned} \|g(t) - h(t)\|^2 &= \|ST(t, g(t)) - TS(t, h(t))\|^2 \\ &\leq \alpha \max\{\|g(t) - ST(t, g(t))\|^2, \|h(t) - TS(t, h(t))\|^2, \|g(t) - TS(t, h(t))\|^2, \\ &\quad \|h(t) - ST(t, g(t))\|^2, \|g(t) - h(t)\|^2\} \quad (\text{by 2.3}) \end{aligned}$$

$$\begin{aligned}
&= \alpha \max\{\|g(t) - g(t)\|^2, \|h(t) - h(t)\|^2, \|g(t) - h(t)\|^2, \|h(t) - g(t)\|^2, \|g(t) - h(t)\|^2\} \\
&\Rightarrow (1 - \alpha)\|g(t) - h(t)\|^2 \leq 0 \\
&\Rightarrow \|g(t) - h(t)\|^2 = 0, \text{ [as } \alpha < \frac{1}{4} \text{ by (2.4)}] \\
&\Rightarrow g(t) = h(t) \text{ for all } t \in \Omega
\end{aligned}$$

This completes the proof of the theorem 3.2.

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# ON GENERALIZED KÖTHE-TOEPLITZ DUALS

HEMEN DUTTA

**ABSTRACT :** In this article we introduce the notion of  $\Phi^\alpha$  - dual, where  $\Phi$  an Orlicz function in order to generalize the notion of Köthe-Toeplitz dual. We prove that  $\Phi^\alpha$  - dual of a sequence space is a  $BK$ -space under a suitable norm. Further we introduce the notion of  $\Phi^\alpha$  - perfect and investigate certain conditions under which two Orlicz functions are equivalent.

**Key words :** Sequence space; Orlicz function; Köthe-Toeplitz dual;  $BK$  space.

**AMS Classification No.:** 40A05, 40C05, 46A45.

## 1. INTRODUCTION

$w$  denotes the space of all scalar sequences and any subspace of  $w$  is called a sequence space.

An Orlicz function is a function  $M: [0, \infty) \rightarrow [0, \infty)$ , which is continuous, non-decreasing and convex with  $M(0) = 0$ ,  $M(x) > 0$ , for  $x > 0$  and  $M(x) \rightarrow \infty$ , as  $x \rightarrow \infty$ .

Two Orlicz functions  $M_1$  and  $M_2$  are said to be equivalent if there are positive constants  $\alpha$ ,  $\beta$  and  $x_0$  such that

$$M_1(\alpha x) \leq M_2(x) \leq M_1(\beta x) \text{ for all } x \text{ with } 0 \leq x \leq x_0.$$

Lindenstrauss and Tzafriri [5] used the Orlica function and introduced the sequence space  $l_M$  as follows:

$$l_M = \left\{ (x_k) \in w: \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

They proved that  $l_M$  is a Banach space normed by

$$\|(x_k)\| = \inf \left\{ \rho > 0: \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

**Lemma 1.** Let  $M_1$  and  $M_2$  be two Orlicz functions. Then  $M_1$  and  $M_2$  are equivalent if and only if  $l_{M_1} = l_{M_2}$ .

The notion of duals of sequence spaces was introduced by Köthe and Toeplitz [4]. Later on it was studied by Kizmaz [2], Bektaş, Et and Çolak [1] and many others.

Let  $E$  and  $F$  be two sequence spaces. Then the  $F$  dual of  $E$  is defined as

$$E^F = \{(x_k) \in w : (x_k y_k) \in F \text{ for all } (y_k) \in E\}.$$

For  $F = l_1$ , the dual is termed as Köthe Toeplitz or  $\alpha$ -dual of  $E$  and denoted by  $E^\alpha$ .  $E$  is said to be  $\alpha$ -perfect if  $E^{\alpha\alpha} = E$ , where  $E^{\alpha\alpha} = (E^\alpha)^\alpha$ .

Let  $E$  be a linear subspace of  $w$ . Then we define the following sequence space for an Orlicz function  $\Phi$ :

$$E_\Phi^\alpha = \left\{ a = (a_k) : \sum_{k=1}^{\infty} \Phi\left(\frac{|a_k x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \text{ and for every } (x_k) \in E \right\}$$

When  $\Phi(x) = x$  for all  $x$  in  $[0, \infty)$ , then  $E_\Phi^\alpha = E^\alpha$ , Köthe-Toeplitz dual of  $E$ . In this connection we call the sequence space  $E_\Phi^\alpha$  as  $\Phi^\alpha$ -dual of  $E$ .

We shall write  $f \approx g$  for non-negative function  $f$  and  $g$  whenever  $C_1 f \leq g \leq C_2 f$  for some  $C_i > 0$ ,  $i = 1, 2$ .

## 2. MAIN RESULTS

**Theorem 2.1.**  $E_\Phi^\alpha$  is a linear space.

**Proof.** Let  $(a_k)$  and  $(b_k)$  be any two elements in  $E_\Phi^\alpha$ . Then there exist some  $\rho_1, \rho_2 > 0$  such that

$$\sum_{k=1}^{\infty} \Phi\left(\frac{|a_k x_k|}{\rho_1}\right) < \infty \text{ and } \sum_{k=1}^{\infty} \Phi\left(\frac{|b_k x_k|}{\rho_2}\right) < \infty \text{ for every } (x_k) \text{ in } E.$$

Let  $\alpha, \beta$  be any scalars and let  $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$ . Then we have

$$\begin{aligned} \sum_{k=1}^{\infty} \Phi\left(\frac{|(\alpha a_k + \beta b_k)x_k|}{\rho_3}\right) &\leq \sum_{k=1}^{\infty} \Phi\left(\frac{|\alpha a_k x_k|}{\rho_3} + \frac{|\beta b_k x_k|}{\rho_3}\right) \\ &\leq \frac{1}{2} \sum_{k=1}^{\infty} \Phi\left(\frac{|a_k x_k|}{\rho_1}\right) + \frac{1}{2} \sum_{k=1}^{\infty} \Phi\left(\frac{|b_k x_k|}{\rho_2}\right) \\ &< \infty, \text{ for every } (x_k) \text{ in } E. \end{aligned}$$

Hence  $E_\Phi^\alpha$  is a linear space.



**Theorem 2.2.**  $E_{\Phi}^{\alpha}$  is a normed linear space under the norm  $\|\cdot\|_{\Phi}^{\alpha}$  defined by

$$\|a\|_{\Phi}^{\alpha} = \inf \left\{ \rho > 0: \sum_{k=1}^{\infty} \Phi \left( \frac{|a_k x_k|}{\rho} \right) \leq 1 \right\} \quad (2.1)$$

**Proof.** It is obvious that if  $a = \theta$ , then  $\|a\|_{\Phi}^{\alpha} = 0$ . Conversely assume that  $\|a\|_{\Phi}^{\alpha} = 0$ . Then using (2.1), we have

$$\inf \left\{ \rho > 0: \sum_{k=1}^{\infty} \Phi \left( \frac{|a_k x_k|}{\rho} \right) \leq 1 \right\} = 0$$

This implies that for a given  $\varepsilon > 0$ , there exists some  $\rho_{\varepsilon}$  ( $0 < \rho_{\varepsilon} < \varepsilon$ ) such that

$$\sum_{k=1}^{\infty} \Phi \left( \frac{|a_k x_k|}{\rho_{\varepsilon}} \right) \leq 1$$

It follows that

$$\Phi \left( \frac{|a_k x_k|}{\rho_{\varepsilon}} \right) \leq 1 \text{ for every } k \leq 1$$

We may choose  $(x_k)$  in  $E$  so that

$$\Phi \left( \frac{|a_k|}{\rho_{\varepsilon}} \right) \leq 1 \text{ for every } k \geq 1$$

and so  $\Phi \left( \frac{|a_k|}{\varepsilon} \right) \leq \Phi \left( \frac{|a_k|}{\rho_{\varepsilon}} \right) \leq 1$  for every  $k \geq 1$ .

This implies that  $a_k = 0$  for all  $k \geq 1$ . Suppose if possible for some  $n$ ,  $a_n$  is non zero, then for this particular  $k$ , we have  $\Phi \left( \frac{|a_n|}{\varepsilon} \right) \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ , a contradiction. Thus  $a = \theta$ .

Let  $(a_k)$  and  $(b_k)$  be any two elements in  $E_{\Phi}^{\alpha}$ . Then there exist some  $\rho_1, \rho_2 > 0$  such that

$$\sum_{k=1}^{\infty} \Phi\left(\frac{|a_k x_k|}{\rho_1}\right) \leq 1 \quad \text{and} \quad \sum_{k=1}^{\infty} \Phi\left(\frac{|b_k x_k|}{\rho_2}\right) \leq 1 \quad \text{for every } (x_k) \text{ in } E.$$

Considering  $\rho = \rho_1 + \rho_2$ , we have

$$\begin{aligned} \sum_{k=1}^{\infty} \Phi\left(\frac{|(a_k + b_k)x_k|}{\rho}\right) &\leq \left(\frac{\rho_1}{\rho_1 + \rho_2}\right) \sum_{k=1}^{\infty} \Phi\left(\frac{|a_k x_k|}{\rho_1}\right) + \left(\frac{\rho_2}{\rho_1 + \rho_2}\right) \sum_{k=1}^{\infty} \Phi\left(\frac{|b_k x_k|}{\rho_2}\right) \\ &\leq \left(\frac{\rho_1}{\rho_1 + \rho_2}\right) + \left(\frac{\rho_2}{\rho_1 + \rho_2}\right) = 1 \end{aligned}$$

Now

$$\begin{aligned} \|a + b\|_{\Phi}^{\alpha} &= \inf \left\{ \rho > 0: \sum_{k=1}^{\infty} \Phi\left(\frac{|(a_k + b_k)x_k|}{\rho}\right) \leq 1 \right\} \\ &\leq \inf \left\{ \rho_1 > 0: \sum_{k=1}^{\infty} \Phi\left(\frac{|a_k x_k|}{\rho_1}\right) \leq 1 \right\} + \inf \left\{ \rho_2 > 0: \sum_{k=1}^{\infty} \Phi\left(\frac{|b_k x_k|}{\rho_2}\right) \leq 1 \right\} \\ &= \|a\|_{\Phi}^{\alpha} + \|b\|_{\Phi}^{\alpha}. \end{aligned}$$

Thus  $\|a + b\|_{\Phi}^{\alpha} \leq \|a\|_{\Phi}^{\alpha} + \|b\|_{\Phi}^{\alpha}$ .

Lastly, let  $\lambda$  be any scalar. Then

$$\begin{aligned} \|\lambda a\|_{\Phi}^{\alpha} &= \inf \left\{ \rho > 0: \sum_{k=1}^{\infty} \Phi\left(\frac{|\lambda a_k x_k|}{\rho}\right) \leq 1 \right\} \\ &= \inf \left\{ (|\lambda| \eta) > 0: \sum_{k=1}^{\infty} \Phi\left(\frac{|a_k x_k|}{\eta}\right) \leq 1 \right\}, \quad \text{where } \eta = \frac{\rho}{|\lambda|} \end{aligned}$$

Thus  $\|\lambda a\|_{\Phi}^{\alpha} = |\lambda| \|a\|_{\Phi}^{\alpha}$ .

This completes the proof.

**Theorem 2.3.**  $E_{\Phi}^{\alpha}$  is a Banach space under the norm  $\|\cdot\|_{\Phi}^{\alpha}$  defined by (2.1).

**Proof.** Let  $(a')$  be any Cauchy sequence in  $E_{\Phi}^{\alpha}$ . Let  $x_0 > 0$  be fixed and  $t > 0$  be such that for  $0 < \varepsilon < 1$ ,  $\frac{\varepsilon}{x_0 t} > 0$ , and  $x_0 t \geq 1$ . Then there exists a positive integer  $n_0$  such that

$$\|a' - a'\|_{\Phi}^{\alpha} < \frac{\varepsilon}{x_0 t} \text{ for all } i, j \geq n_0.$$

Using (2.1), we have

$$\|a' - a'\|_{\Phi}^{\alpha} = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} \Phi \left( \frac{|(a'_k - a'_k)x_k|}{\rho} \right) \leq 1 \right\} < \frac{\varepsilon}{x_0 t} \text{ for all } i, j \leq n_0$$

This implies that

$$\sum_{k=1}^{\infty} \Phi \left( \frac{|(a'_k - a'_k)x_k|}{\|a' - a'\|_{\Phi}^{\alpha}} \right) \leq 1 \text{ for all } i, j \geq n_0$$

It follows that

$$\Phi \left( \frac{|(a'_k - a'_k)x_k|}{\|a' - a'\|_{\Phi}^{\alpha}} \right) \leq 1 \text{ for all } i, j \geq n_0 \text{ and for all } k \geq 1.$$

Since  $E$  is a subspace of  $w$ , we may take  $(x_k) = \{1, 1, 1, \dots\}$ . Hence we have

$$\Phi \left( \frac{|a'_k - a'_k|}{\|a' - a'\|_{\Phi}^{\alpha}} \right) \leq 1 \text{ for all } i, j \geq n_0 \text{ and for all } k \geq 1$$

For  $t > 0$  with  $\Phi\left(\frac{tx_0}{2}\right) \geq 1$ , we have

$$\Phi\left(\frac{|a'_k - a'_j|}{\|a' - a'\|_\Phi^\alpha}\right) \leq \Phi\left(\frac{tx_0}{2}\right) \text{ for all } i, j \geq n_0 \text{ and for all } k \geq 1$$

Since  $\Phi$  is non-decreasing, we have

$$|a'_k - a'_j| \leq \frac{tx_0}{2} \|a' - a'\|_\Phi^\alpha \leq \frac{tx_0}{2} \cdot \frac{\varepsilon}{x_0 t} = \frac{\varepsilon}{2} \text{ for all } i, j \geq n_0 \text{ and for all } k \geq 1.$$

Thus  $(a'_k)$  is a Cauchy sequence in  $C$  for all  $k \geq 1$  and so it is convergent in  $C$  for all  $k \geq 1$ .

Let  $\lim_{i \rightarrow \infty} a'_k = a_k$  for all  $k \geq 1$ .

For  $i, j \geq n_0$ , we have

$$\inf \left\{ \rho > 0: \sum_{k=1}^{\infty} \Phi \left( \frac{|(a'_k - a'_j)x_k|}{\rho} \right) \leq 1 \right\} < \varepsilon$$

Since  $\Phi$  is continuous we have

$$\inf \left\{ \rho > 0: \sum_{k=1}^{\infty} \Phi \left( \frac{|(a'_k - a_k)x_k|}{\rho} \right) \leq 1 \right\} < \varepsilon, \text{ for } i \geq n_0 \text{ and as } j \rightarrow \infty$$

It follows that  $(a' - a) \in E_\Phi^\alpha$ . Since  $(a') \in E_\Phi^\alpha$  and  $E_\Phi^\alpha$  is linear  $a = (a_k) \in E_\Phi^\alpha$ . Hence every Cauchy sequence in  $E_\Phi^\alpha$  converges to an element of  $E_\Phi^\alpha$ . Thus  $E_\Phi^\alpha$  is Banach space.

**Remark.** From the above proof we observe that  $E_\Phi^\alpha$  is always a Banach space whether  $E$  is a Banach space or not.

From the above proof we can easily conclude that  $\|a' - a\|_{\Phi}^{\alpha} \rightarrow 0$  implies  $|a'_k - a_k| \rightarrow 0$  as  $i \rightarrow \infty$ . Hence we have the following Theorem.

**Theorem 2.4.**  $E_{\Phi}^{\alpha}$  is a BK space under the norm  $\|\cdot\|_{\Phi}^{\alpha}$  defined by (2.1).

**Theorem 2.5.** (i)  $\Phi \subset E_{\Phi}^{\alpha}$

(ii) If  $E \subset F$ , then  $F_{\Phi}^{\alpha} \subset E_{\Phi}^{\alpha}$

(iii)  $E \subset D_{\Phi}^{\alpha}$ , where  $D = E_{\Phi}^{\alpha}$ .

**Proof.** Proof is easy and so omitted.

In view of the above Theorem we give the following definition:

'A sequence space  $E$  is said to be a  $\Phi^{\alpha}$ -perfect if  $D_{\Phi}^{\alpha} = E$ , where  $D = E_{\Phi}^{\alpha}$ '.

**Theorem 2.6.** Let  $M$  be a sequence of Orlicz functions. If  $\lim_{t \rightarrow 0} \frac{M(t)}{t} > 0$  and  $\lim_{t \rightarrow 0} \frac{M(t)}{t} < \infty$  then  $E_{\Phi}^{\alpha} = E^{\alpha}$ .

**Proof.** If the given conditions are satisfied, we have  $M(t) \approx t$ . Hence the proof follows.

**Theorem 2.7.** Let  $M$  and  $N$  be two equivalent Orlicz functions. Then  $E_M^{\alpha} = E_N^{\alpha}$ .

**Proof.** Proof is obvious and so omitted.

Taking  $\Phi(x) = x$  in the definition of  $E_{\Phi}^{\alpha}$ , we can compute Köthe-Toeplitz dual of  $E$ . (See [3]). Our next aim is to obtain  $\Phi^{\alpha}$ -duals for those  $\Phi$ 's, which satisfies the additional property  $\Phi(x, y) = \Phi(x) \cdot \Phi(y)$ , for every  $x, y$  in  $[0, \infty)$ ... (2.2)

**Theorem 2.8.** Let  $M$  be any Orlicz function. Then

(i)  $[l_M]_{\Phi}^{\alpha} = l_{\Phi}$ ,

(ii)  $[l_{\Phi}]_{\Phi}^{\alpha} = l_{\Phi}$

(iii)  $L_{\Phi}$  is  $\Phi^{\alpha}$ -perfect.

**Proof.** (i) Let  $a \in l_\Phi$ . Then  $\sum_{k=1}^{\infty} \Phi\left(\frac{|a_k|}{\rho}\right) < \infty$  for some  $\rho > 0$ . Now for any  $x \in l_M$ , we have

$$\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho_1}\right) < \infty \text{ for some } \rho_1 > 0 \text{ and so } (x_k) \in l_\infty. \text{ Hence } \sup_k \Phi(|x_k|) < \infty.$$

Now we have

$$\sum_{k=1}^{\infty} \Phi\left(\frac{|a_k x_k|}{\rho}\right) = \sum_{k=1}^{\infty} \Phi\left(\frac{|a_k|}{\rho}\right) \Phi(|x_k|) \leq \sup_k \Phi(|x_k|) \sum_{k=1}^{\infty} \Phi\left(\frac{|a_k|}{\rho}\right) < \infty$$

Hence  $a \in [l_M]_\Phi^\alpha$ .

Conversely suppose that  $a \in [l_M]_\Phi^\alpha$ , then  $\sum_{k=1}^{\infty} \Phi\left(\frac{|a_k x_k|}{\rho}\right) < \infty$  for some  $\rho > 0$  and every

$x \in l_M$ . Suppose if possible  $a \notin l_\Phi$ . Then there exists a strictly increasing sequence  $(n_i)$  of positive integers  $n_i$  with  $n_1 < n_2 < \dots$ , such that

$$\sum_{k=n_i+1}^{n_{i+1}} \Phi\left(\frac{|a_k|}{\rho}\right) < \Phi(i), \text{ for every } \rho > 0.$$

Define  $x \in l_M$  by

$$\begin{aligned} x_k &= 0, \quad 1 \leq k \leq n_1 \\ &= \operatorname{sgn} a_k / i, \quad n_i < k \leq n_{i+1}. \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{k=1}^{\infty} \Phi\left(\frac{|a_k x_k|}{\rho}\right) &= \sum_{k=1}^{\infty} \Phi\left(\frac{|a_k|}{\rho}\right) \Phi(|x_k|) \\ &= \sum_{k=n_1+1}^{n_2} \Phi\left(\frac{|a_k|}{\rho}\right) \Phi(|x_k|) + \dots + \sum_{k=n_i+1}^{n_{i+1}} \Phi\left(\frac{|a_k|}{\rho}\right) \Phi(|x_k|) + \dots \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=n_1+1}^{n_2} \Phi\left(\frac{|a_k|}{\rho}\right) \Phi(1) + \dots + \sum_{k=n_i+1}^{n_{i+1}} \Phi\left(\frac{|a_k|}{\rho}\right) \Phi\left(\frac{1}{i}\right) + \dots \\
&> \Phi(1) + \dots + \Phi(1) + \dots = \infty
\end{aligned}$$

This contradicts to  $a \in [l_M]_{\Phi}^{\alpha}$ . Hence  $a \in l_{\Phi}$ . This completes the proof of (i).

(ii) Proof follows from part (i) by taking  $\Phi$  instead of  $M$ .

(iii) Proof follows from part (ii).

**Theorem 2.9.** *For any Orlicz function  $M$ ,  $l_M$  is  $\Phi^{\alpha}$ -perfect if and only if  $M$  and  $\Phi$  are equivalent.*

**Proof.** First assume  $l_M$  is  $\Phi^{\alpha}$ -perfect. Then by Theorem 2.8(ii),  $l_M = l_{\Phi}$  and so  $M$  and  $\Phi$  are equivalent.

Conversely assume  $M$  and  $\Phi$  are equivalent. Then  $l_M = l_{\Phi}$  and so  $l_M$  is  $\Phi^{\alpha}$ -perfect by Theorem 2.8.

**Theorem 2.10.** *Let  $M$  and  $N$  be two Orlicz functions which satisfy the additional condition (2.2). Then  $M$  and  $N$  are equivalent if and only if  $l_M$  is  $N^{\alpha}$ -perfect or  $l_N$  is  $M^{\alpha}$ -perfect.*

**Proof.** Proof follows from Theorem 2.9.

Our next Theorem is guided by the fact that every Orlicz sequence space has a subspace isomorphic to  $l^p$  ( $1 \leq p < \infty$ ).

**Theorem 2.11.** (i) *Every Orlicz sequence space has a subspace which is  $[l^p]^{\alpha}$ -perfect, where  $1 \leq p < \infty$ .*

(ii) *Every Orlicz sequence space has a subspace  $U$  such that  $l^p$  ( $1 \leq p < \infty$ ) is  $U^{\alpha}$ -perfect.*

**Proof.** Proof follows from Theorem 2.10.

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# A MINIMAL POINT THEOREM IN PRODUCT SPACES

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**ABSTRACT :** We derive a minimal point theorem, for a subset  $A$  in a cone in product space under a weak assumption concerning the boundedness of the considered set  $A$ . Using the result we improve the vectorial variant of Ekeland's variational principle. We prove the result for a Product space  $X \times Y$ , where  $X$  is a  $T_0$  quasi-gauge topological space, generated by a family of quasi-pseudo metrics and  $Y$  is a separated, locally convex space.

**Key words :** Quasi-gauge space, Ekeland's variational principle, minimal points Cone-valued metric.  
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## 1. INTRODUCTION

Phelps observed that Ekeland's variational principle (EVP) is equivalent to the existence of a minimal point of the epigraph of the corresponding function with respect to an appropriate order. Several authors such as Nemeth, Isac and Dentcheva and Helbig have generalized the EVP for vector-valued functions. Nemeth obtained a vectorial Ekeland's variational principle using cone-valued metrics. Gopfert and Tamer [12], [13], established several product spaces and the corresponding variants of the vectorial Ekeland's variational principle.

In this extension, we derive a minimal point theorem, for a subset  $A$  in a cone in a product space under a weak assumption concerning the boundedness of the considered set  $A$ . Using the result we improve the vectorial variant of Ekeland's variational principle.

We prove the result for a Product space  $X \times Y$ , where  $X$  is a  $T_0$  quasi-gauge topological space, generated by a family of quasi-pseudo metrics and  $Y$  is a separated, locally convex space.

## 2. QUASI-GAUGE SPACE

a quasi-pseudo metric on a set  $X$  is a non-negative real valued function on  $X \times X$  such that for any  $x, y, z \in X$ .

$$p(x, x) = 0 \text{ for all } x \in X \text{ and } p(x, y) \leq (p(x, z) + p(z, y))$$

A quasi-gauge structure for a topological space is a family  $P$  of quasi-pseudo metrics

on  $X$  such that  $T$  has a sub-base, the family  $\{B(x, p, \varepsilon) : x \in X, p \in P, \varepsilon > 0\}$  where  $B(x, p, \varepsilon)$  is the set  $\{y \in X : p(x, y) < \varepsilon\}$ . If a topological space  $(X, T)$  has a quasi-gauge structure  $P$ , then it is called a quasi-gauge space. Every topological space is a quasi-gauge space. The sequence  $\{x_n\}$  in  $X$  is called left (right)  $P$ -Cauchy sequence, if for each  $p$  in  $P$ , and each  $\varepsilon > 0$ , there is a point  $x$  in  $X$  and an integer  $k$  such that

$$p(x, x_m) < \varepsilon \text{ [} p(x_m, x) < \varepsilon \text{] for all } m \geq k. (x \text{ and } k \text{ may depend on } \varepsilon \text{ and } p).$$

A quasi-gauge space  $(X, P)$  is left (right)  $P$ -sequentially complete, if every left (right)  $P$ -Cauchy sequence in  $X$  converges to some element of  $X$ .  $X$  is a  $T_0$  space iff

$$p(x, y) = p(y, x) = 0 \text{ for all } p \text{ in } P \text{ implies } x = y.$$

$$X \text{ is a } T_1 \text{ space iff } p(x, y) = 0 \text{ for all } p \text{ in } P \text{ implies } x = y.$$

### 3. EKELAND'S VARIATIONAL PRINCIPLE

Let  $X$  be a complete metric space. Let  $\delta > 0$  and  $u \in X$  be given such that  $\phi : X \rightarrow R \cup \{+\infty\}$  is a lower semi continuous function and bounded below.

Let  $\varepsilon > 0$  and  $u_1 \in X$  be given such that

$$\phi(u_1) \leq \inf_x \phi + \varepsilon/2 \text{ and then given } \lambda > 0 \text{ there exists } u_\lambda \in X \text{ such that}$$

$$\phi(u_\lambda) \leq \phi(u_1); \quad d(u_\lambda, u_1) \leq \lambda$$

$$\phi(u_\lambda) < \phi(u) + \varepsilon/\lambda \quad d(u_\lambda, u) \text{ for all } u \neq u_\lambda.$$

$(X, P)$  is a seq. complete quasi-gauge space,  $Y$  is a separated, locally convex space.  $Y^*$  is its topological dual.  $K \subset Y$  is a convex cone. i.e.  $K + K \subset K$  and  $[0, \infty) \cdot K \subset K$

$$K^+ = \{y^* \in Y^* \mid \langle y, y^* \rangle \geq 0 \quad \forall y \in K\} \text{ is the dual cone of } K \text{ and}$$

$$K^\# = \{y^* \in Y^* \mid \langle y, y^* \rangle > 0 \quad \forall y \in K \setminus \{0\}\}$$

In this we suppose that  $K$  is pointed i.e.  $K \cap (-K) = \{0\}$

The cone  $K$  determines an order relation on  $Y$ , denoted in the sequel by ' $\leq_k$ ': so for  $y_1, y_2 \in Y$ ,  $y_1 \leq_k y_2$ , if  $y_2 - y_1 \in K$ . It is reflexive, transitive and anti symmetric. Let  $k^0 \in K \setminus \{0\}$ . Using  $k^0$  we introduce an order relation on  $X \times Y$  denoted by ' $\leq_k^0$ '.

$$(x_1, y_1) \leq_k^0 (x_2, y_2) \text{ if } y_1 + k^0 p(x_1 - x_2) \leq_k y_2 \quad \forall p \in P$$

Then ' $\leq_k^0$ ' is reflexive, transitive and anti symmetric. That is our notation are as in those of [9].

For the derivation of the minimal point theorem, we make use of larger cone  $B \subset Y$  (to define a suitable functional  $z_B : Y \rightarrow R$ ) including the ordering cone  $K \subset Y$ ;  $K \setminus \{0\} \subset \text{int } B$ . also we will replace the usual boundedness condition of the projection  $P_Y A$  of  $A$  onto  $Y$  by a weaker one. Moreover, we replace the complete metric space  $X$  by a sequentially complete  $T_0$  or  $T_1$  space.

**Theorem 1.** Assume that there exists a proper convex cone  $B \subset Y$  such that  $K \setminus \{0\} \subset \text{int } B$ . Suppose that the set  $A \subset X \times Y$  satisfies the conditions

$(H_1)$  for every  $\leq_k^0$  - decreasing sequence  $((x_n, y_n)) \subset A$  with  $x_n \rightarrow x \in X$

There exists  $y \in Y$  such that  $(x, y) \in A$  and  $(x, y) \leq_k^0 (x_n, y_n)$  for every  $n \in N$  and that  $P_Y(A) \cap (\dot{y} - \text{int } B) = \emptyset$  for some  $\dot{y} \in Y$ . Then for every  $(x_0, y_0) \in A$ , there exists  $(x^-, y^-) \in A$ , minimal with respect to  $\leq_k^0$  such that  $(x^-, y^-) \leq_k^0 (x_0, y_0)$ .

**Proof.** Let

$$z_B : Y \rightarrow R, z_B(y) = \inf\{t \in R \mid y \in tk^0 - cl B\}$$

By lemma 7 of (Nemeth A.B:), ' $z_B$ ' is a continuous sub linear function such that

$$z_B(y + tk^0) = z_B(y) + t \text{ for all } t \in R \text{ and } y \in Y, \text{ and for every } \lambda \in R.$$

$$\{y \in Y : z_B(y) \leq \lambda\} = \lambda k^0 - cl B$$

$$\{y \in Y : z_B(y) < \lambda\} = \lambda k^0 - \text{int } B$$

Moreover if  $y_2 - y_1 \in K \setminus \{0\}$ , then  $z_B(y_1) < z_B(y_2)$ . Observe that for  $(x, y) \in A$ , we have that  $z_B(y - \dot{y}) \geq 0$ . Otherwise for some  $(x, y) \in A$ , we have  $z_B(y - \dot{y}) < 0$ . It follows that there exists  $\lambda > 0$  such that  $y - \dot{y} \in -\lambda k^0 - cl B$ . Hence

$$y \in \dot{y} - (\lambda k^0 + cl B) \subset \dot{y} - (\text{int } B + cl B) \subset \dot{y} - \text{int } B$$

which is a contradiction. Since  $0 \leq z_B(y - \dot{y}) \leq z_B(y) + z_B(\dot{y})$ . It follows that  $z_B$  is bounded from below on  $P_Y(A)$ . Let us construct a sequence  $((x_n, y_n))_{n \geq 0} \subset A$  as follows: having  $(x_n, y_n) \in A$  choose  $(x_{n+1}, y_{n+1}) \in A$  such that  $(x_{n+1}, y_{n+1}) \leq_k^0 (x_n, y_n)$  such that

$$z_B(y_{n+1}) \leq \inf\{z_B(y) \mid (x, y) \in A \text{ and } (x, y) \leq_k^0 (x_n, y_n)\} + 1/n + 1$$

Of course the sequence  $((x_n, y_n))$  is  $\leq_k^0$  - decreasing. It follows that

$$y_{m+p} + k^0 p(x_{m+q}, x_n) \leq_K y_m \quad \forall m, q \in N$$

$$z_B(y_{n+p} + k^0 p(x_{n+p}, x_n)) = z_B(y_{n+p}) + p(x_{n+p}, x_n) \leq z_B(y_n)$$

$$p(x_{n+p}, x_n) \leq z_B(y_n) - z_B(y_p) \leq 1/n \quad \forall n, p \in N \quad \forall p \in P$$

It follows that  $(x_n)$  is a left  $P$  Cauchy sequence in the sequentially complete To quasi-gauge space  $(X, P)$ , and so  $\{x_p\}$  is convergent to some  $x \in X$ . By condition  $H_1$ , there exists  $y \in Y$  such that  $(x, y) \in A$  and  $(x, y) \leq_k^0 (x_n, y_n)$  for every  $n \in N$ .

Let us now show that  $(x, y)$  is the desired element

Indeed  $(x, y) \leq_k^0 (x_0, y_0)$ . Suppose  $(x^1, y^1) \in A$  is such that  $(x^1, y^1) \leq_k^0 (x, y) (\leq_k^0 (x_n, y_n))$  for every  $n \in N$ . Thus  $z_B(y^1) + p(x^1, x) \leq z_B(y)$ , hence

$$p(x^1, x) \leq z_B(y) - z_B(y^1) \leq z_B(y_n) - z_B(y^1) \leq 1/n \quad \forall n \geq 1.$$

It follows that  $p(x^1, x) = z_B(y) - z_B(y^1) = 0 \quad \forall p \in P$

Hence  $x^1 = x$ . Since  $X$  is a  $T_1$  space [the result holds for  $T_0$  space, if we define the order relation in such a way that  $(x_1, y_1) \leq_k^0 (x_2, y_2)$  if  $y_1 + \max(p(x_2, x_2), p(x_2, x_1)) \leq_K y_2$ ]

As  $y^1 \leq_K y$  - if  $y^1 \neq y$   $y - y^1 \in K \setminus \{0\}$ . Hence  $z_B(y^1) < z_B(y)$  which is a contradiction. Hence  $(x^1, y^1) = (x, y)$ . The result is for a - To quasi-gauge space  $X$ , with the ordering on  $X \times Y$  by  $(x_1, y_1) \leq_k^0 (x_2, y_2)$  if  $y_1 + k^0 \max(p(x_1, x_2), p(x_2, x_1)) \leq_K y_2$  and  $T_1$  sequentially complete quasi-gauge space with the ordering defined by  $(x_1, y_1) \leq_k^0 (x_2, y_2)$ , if  $y_1 + k^0 p(x_1, x_2) \leq_K y_2$

The above result for products space, where is a complete metric space is given in [4]. Note that the condition on  $K$  is stronger here - in comparison with (theorem 4). (Since, in this case  $K^* \neq \emptyset$ ) while the condition on  $A$  is weaker. ( $A$  may not be contained in the half space), Further when  $K$  and  $k^0$  are as in theorem 1, corollaries 2 and 3 from [5, 9] may be improved.

In the following corollary, we derive a variational principle of Ekeland's type for objective functions which takes values in a general space  $Y$ ,  $Y^* = Y \cup \{\infty\}$ , with  $\infty \notin Y$ . We consider  $y \leq \infty$  for every  $y \in Y$ . We consider also a function  $f: X \rightarrow Y$  and  $\text{dom } f = \{x \in X; f(x) \neq \infty\}$

**Corollary 2.** Let  $f: X \rightarrow Y^*$ , Assume that there exists a proper convex cone  $B \subset Y$  such that,  $K \setminus \{0\} \subset \text{int } B$  and  $f(x) \cap (y - B) = \emptyset$  for some  $y \in Y$  Also suppose that,

$$H_3\{x^1 \in X/f(x^1) + p(x^1, x) \leq f(x)\} \text{ is closed for every } x \in X \text{ or}$$

$H_4$  for every sequence  $(x_n) \subset \text{dom} f$  with  $x_n \rightarrow x$  and  $(f(x_n)) \leq_k$  decreasing  $f(x) \leq_K f(x_n)$  for every  $n \in N$  and this is closed in the direction of  $k^0$ , then for every  $x_0 \in \text{dom} f$  there exists  $x^- \in X$  such that

$$f(x^-) + k^0 p(x^-, x_0) \leq_K f(x_0)$$

$$\text{and } \forall x \in X, f(x) + k^0 p(x^-, x) \leq f(x^-) \Rightarrow x = x^-$$

We say  $K$  is closed in the direction  $k^0$  if  $K \cap (y - R_+ k^0)$  is closed for every  $y \in K$ .

The proof of corollary 2 is similar to those of corollary 2 & 3 of (9).

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